ELEMENTARY MATHEMATICS

W W L CHEN and X T DUONG

Chapter 8

ELEMENTARY COUNTING TECHNIQUES

8.1. The Fundemental Principle of Counting

We begin by studying two very simple examples.

EXAMPLE 8.1.1. Consider the collection of all 2-digit numbers where the first digit is either 1 or 2, and where the second digit is either 6, 7 or 8. Clearly there are 6 such numbers and they are listed below:

16	17	18
26	27	28

Arranged this way, we note that each row corresponds to a choice for the first digit and each column corresponds to a choice for the second digit. We have 2 rows and 3 columns, and hence $2 \times 3 = 6$ possibilities.

EXAMPLE 8.1.2. Consider the collection of all 3-digit numbers where the first digit is either 1, 2, 3 or 4, where the second digit is either 5 or 6, and where the third digit is either 7, 8 or 9. The candidates are listed below:

157	158	159	257	258	259	357	358	359	457	458	459
167	168	169	267	268	269	367	368	369	467	468	469

Arranged this way, we note that each block corresponds to a choice for the first digit. Within each block, each row corresponds to a choice for the second digit and each column corresponds to a choice for the third digit. We have 4 blocks, each with 2 rows and 3 columns, and hence $4 \times 2 \times 3 = 24$ possibilities.

These two examples are instances of a simple but very useful principle.

† This chapter was written at Macquarie University in 1999.

FUNDAMENTAL PRINCIPLE OF COUNTING. *Suppose that a first event can occur in n_1 different ways, a second event can occur in n_2 different ways, and so on, and a k-th event can occur in n_k different ways. Then the number of different ways for these k events to occur in succession is given by $n_1 \times n_2 \times \ldots \times n_k$.*

Example 8.1.3. Consider motor vehicle licence plates made up of 3 letters followed by 3 digits, such as *ABC012*. To determine the total number of possible different licence plates, note that there are 26 choices for each letter and 10 choices for each digit. Hence the total number is $26 \times 26 \times 26 \times 10 \times 10 \times 10$. On the other hand, if the first digit is restricted to be non-zero, then the total number is only $26 \times 26 \times 26 \times 9 \times 10 \times 10$. Furthermore, if the letters are required to be distinct and the first digit is restricted to be non-zero, then the total number is only $26 \times 25 \times 24 \times 9 \times 10 \times 10$. Finally, if the letters and digits are required to be distinct and the first digit is restricted to be non-zero, then the total number is only $26 \times 25 \times 24 \times 9 \times 9 \times 8$.

8.2. Permutation

Again, we begin by studying a simple example.

Example 8.2.1. Suppose that we are required to choose 6 people out of 10 and seat them in a row from the left to the right. To do this, let us take a sensible approach and choose them one at a time. We can fill the leftmost seat by any one of the 10 people, so we clearly have 10 choices here. Having made a choice, we can fill the second seat by any one of the 9 remaining people, so we clearly have 9 choices here. Having made a choice, we can fill the third seat by any one of the 8 remaining people, so we clearly have 8 choices here, and so on. Clearly the total number of ways is equal to

$$P(10,6) = \underbrace{10 \times 9 \times 8 \times 7 \times 6 \times 5}_{6}.$$

PERMUTATION. *The number of ways of choosing k objects from n objects and arranging them in order is equal to*

$$P(n,k) = \underbrace{n \times (n-1) \times \ldots \times (n-k+1)}_{k} = \frac{n!}{(n-k)!}.$$

Here, for every positive integer m, the number $m!$ denotes the product of the first m positive integers. We shall also use the convention that $0! = 1$.

Example 8.2.2. We wish to determine the number of 5-digit numbers where all the digits are non-zero and where at least one digit is used more than once. To do so, we formulate the following strategy. Our desired number is equal to $N_1 - N_2$, where N_1 denotes the total number of 5-digit numbers where all the digits are non-zero, and where N_2 denotes the total number of 5-digit numbers where all the digits are non-zero and where no digit is used more than once. Then

$$N_1 = 9 \times 9 \times 9 \times 9 \times 9 = 59049 \qquad \text{and} \qquad N_2 = P(9,5) = 9 \times 8 \times 7 \times 6 \times 5 = 15120,$$

so that $N_1 - N_2 = 43929$.

Example 8.2.3. We wish to determine the number of ways of arranging 6 people in a circle. To do this, we put 6 chairs in a circle and ask person A to choose a chair. Clearly it does not matter which chair he chooses. It is only a question of which chairs the others choose relative to him. So he really has only one choice. Now person B has 5 chairs to choose from, and person C has 4 chairs to choose from, and so on, while person F takes whatever is left. The total number of ways is therefore equal to $P(5,5) = 120$. Alternatively, we nominate one of these chairs and call it the first chair. Then we fill the chairs in clockwise order. We can put any one of the 6 people in the first chair, any one of the remaining 5 in the second chair, and so on. Hence there are $P(6,6)$ ways of achieving these. Now note that the following two arrangements are the same:

$$\begin{array}{ccccccccc} & A & B & & \qquad & & F & A & \\ F & & & C & \qquad & E & & & B \\ & E & D & & \qquad & & D & C & \end{array}$$

Indeed, there are 4 more arrangements which are the same as these two. Hence we have overcounted by a multiple of 6, and so the desired number is

$$\frac{1}{6} \times P(6,6) = 120$$

as before.

EXAMPLE 8.2.4. We wish to determine the number of ways of arranging the letters of

$$WOOLLOOMOOLOO.$$

Here there are 8 O's, 3 L's and one each of W and M, making a total of 13 letters. First of all, let us label the O's and L's to make them different, so that

$$MO_1O_2L_aL_bO_3O_4WO_5O_6L_cO_7O_8 \qquad \text{and} \qquad MO_2O_1L_aL_bO_3O_4WO_5O_6L_cO_7O_8$$

are considered to be different arrangements. In this case, there are clearly $P(13,13) = 13!$ arrangements. On the other hand, the 8 different O's are having a private contest among themselves to see who is used first, second, and so on, and they have $P(8,8) = 8!$ ways of resolving this. Similarly, the 3 different L's have $P(3,3) = 3!$ ways of resolving their own little dispute. Unlabelling the O's and L's, we see that we have overcounted, and the correct number is really only

$$\frac{13!}{8!3!} = 25740.$$

EXAMPLE 8.2.5. We shall make 5-digit numbers from the digits $1,2,3,4,5,6,7,8$, where no digit is used more than once.

- If there are no further restrictions, then clearly there are $P(8,5)$ ways.
- Suppose that the 5-digit number must begin with the digit 1. Then there are clearly $P(7,4)$ ways of choosing the remaining 4 digits.
- Suppose that the 5-digit number must contain the digit 1. Then there are $P(5,1)$ choices for placing the digit 1. Having made this choice, there are $P(7,4)$ ways of choosing the remaining 4 digits. Hence the total number of ways is equal to $P(5,1) \times P(7,4)$.
- Suppose that the 5-digit number must begin with the digit 1 and contain the digit 2. Then there are $P(4,1)$ choices for placing the digit 2. Having made this choice, there are $P(6,3)$ ways of choosing the remaining 3 digits. Hence the total number of ways is equal to $P(4,1) \times P(6,3)$.
- Suppose that the 5-digit number must contain the digits 1 and 2. Then there are $P(5,2)$ choices for placing the digits 1 and 2. Having made this choice, there are $P(6,3)$ ways of choosing the remaining 3 digits. Hence the total number of ways is equal to $P(5,2) \times P(6,3)$.

8.3. Combination

Yet again, we begin by studying a simple example.

EXAMPLE 8.3.1. Suppose that we are required to choose 6 people out of 10 but not in any particular order. To do this, let us first return to Example 8.2.1 where 6 people are chosen in order and seated from left to right. In this case, there are $P(10,6)$ choices. However, if the order is no longer important, the 6 chosen people can resolve their differences, and the number of duplications which end up with the same 6 people is the number of ways of arranging 6 people in order, equal to $P(6,6)$. It follows that the number of ways of choosing 6 people out of 10 but not in any particular order is equal to

$$\frac{P(10,6)}{P(6,6)}.$$

COMBINATION. *The number of ways of choosing k objects from n objects not in any particular order is equal to*

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{P(k,k)} = \frac{n!}{k!(n-k)!}.$$

EXAMPLE 8.3.2. Consider the set $S = \{a,b,c,d,e,f,g,h,i\}$ of 9 letters. Suppose that we wish to choose 4 letters from S.

- If there are no restrictions, then there are clearly $C(9,4)$ ways.
- Suppose that we must choose 2 vowels and 2 consonants. Then there are $C(3,2)$ ways of choosing the vowels and $C(6,2)$ ways of choosing the consonants. Hence the total number of ways is equal to $C(3,2) \times C(6,2)$.
- Suppose that we must choose at least one vowel. Then we note that there are $C(6,4)$ ways of choosing all consonants. Hence the desired number of ways is equal to $C(9,4) - C(6,4)$.
- Suppose that we must choose more consonants than vowels. Then we can either choose no vowels or one vowel. If we choose no vowels, then there are $C(6,4)$ possibilities. If we choose one vowel, then there are $C(3,1) \times C(6,3)$ possibilities. Hence the desired number of ways is equal to

$$C(6,4) + C(3,1) \times C(6,3).$$

EXAMPLE 8.3.3. Some crazy professor has set an examination paper which contains 5 history questions and 6 engineering questions. An unfortunate candidate is required to choose exactly 8 questions.

- If there are no further restrictions, then the number of ways is $C(11,8)$.
- Suppose that the unfortunate candidate is required to choose exactly 4 history questions and 4 engineering questions. Then there are $C(5,4)$ ways of choosing the history questions and $C(6,4)$ ways of choosing the engineering questions. Hence the total number of ways is equal to

$$C(5,4) \times C(6,4).$$

- Suppose that the unfortunate candidate is required to choose at least 4 history questions. Then apart from the $C(5,4) \times C(6,4)$ ways of choosing exactly 4 history questions and 4 engineering questions, there are also $C(5,5) \times C(6,3)$ ways of choosing exactly 5 history questions and 3 engineering questions. Hence the total number of ways is equal to

$$C(5,4) \times C(6,4) + C(5,5) \times C(6,3).$$

- Suppose that the unfortunate candidate is required to choose at most 4 history questions. Then apart from the $C(5,4) \times C(6,4)$ ways of choosing exactly 4 history questions and 4 engineering questions, there are also $C(5,3) \times C(6,5)$ ways of choosing exactly 3 history questions and 5 engineering questions, as well as $C(5,2) \times C(6,6)$ ways of choosing exactly 2 history questions and 6 engineering questions. Hence the total number of ways is equal to

$$C(5,4) \times C(6,4) + C(5,3) \times C(6,5) + C(5,2) \times C(6,6).$$

Alternatively, the only way of choosing 8 questions and falling foul of the crazy professor is to choose all 5 history questions and 3 engineering questions. There are $C(5,5) \times C(6,3)$ ways of doing so. Hence the total number of ways is also equal to $C(11,8) - C(5,5) \times C(6,3)$.

EXAMPLE 8.3.4. Consider a usual deck of 52 playing cards, without jokers. We wish to choose 5 cards from this deck.

- Without any restrictions, the number of ways is equal to $C(52,5)$.
- Suppose that we wish to choose exactly two A's and three K's. Then the total number of ways is equal to $C(4,2) \times C(4,3)$.
- Suppose that we wish to have four cards of the same suit and a fifth card from another suit. There are $C(4,1)$ ways of choosing a suit, and then $C(13,4)$ ways of choosing four cards from it. There are then $C(39,1)$ ways of choosing the fifth card. Hence the total number of ways is equal to $C(4,1) \times C(13,4) \times C(39,1)$.
- Suppose that we wish to choose four cards with the same value and another card. There are $C(13,1)$ ways of choosing a value and then $C(4,4)$ ways of choosing the four cards with this value. There are then $C(48,1)$ ways of choosing the remaining card. Hence the total number of ways is equal to $C(13,1) \times C(4,4) \times C(48,1)$.
- Suppose that the five cards we choose must have total value equal to 7. Here let us assume that A's carry the value 1. Then the only possibilities are four A's and one 3, or three A's and two 2's. The number of choices for the former is $C(4,4) \times C(4,1)$, while the number of choices for the latter is $C(4,3) \times C(4,2)$. Hence the total number is equal to $C(4,4) \times C(4,1) + C(4,3) \times C(4,2)$.

8.4. The Binomial Theorem

Consider the array of numbers given below:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\end{array}$$
$$\begin{array}{ccc} \iddots & \qquad\qquad\qquad & \ddots \end{array}$$

This is known as Pascal's triangle. It has the property that every entry is equal to the sum of the two immediate entries on the row above. In fact, Pascal's triangle can be written in the following form:

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\end{array}$$
$$\begin{array}{ccc} \iddots & \qquad\qquad\qquad & \ddots \end{array}$$

In this case, the rule concerning an entry being equal to the sum of the two immediate entries on the row above translates to the observation that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

which can easily be established from the definition. Indeed, we have

$$\binom{n}{k-1}+\binom{n}{k}=\frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}=\frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1}+\frac{1}{k}\right)$$
$$=\frac{n!}{(k-1)!(n-k)!}\times\frac{n+1}{(n-k+1)k}=\frac{(n+1)!}{k!(n+1-k)!}=\binom{n+1}{k}.$$

An important deduction from this observation is the following result.

BINOMIAL THEOREM. *Suppose that $n\in\mathbb{N}$. Then for any real numbers $a,b\in\mathbb{R}$, we have*

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

REMARKS. (1) Note the special case $n=2$, where we obtain

$$(a+b)^2=a^2+2ab+b^2.$$

(2) Note also the special case when $a=1$ and $b=x$. We have

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k.$$

In other words, the binomial coefficient

$$\binom{n}{k}$$

is equal to the coefficient of x^k in the expansion of $(1+x)^n$.

EXAMPLE 8.4.1. Let $a=b=1$. Then the binomial theorem gives

$$2^n=\sum_{k=0}^{n}\binom{n}{k}.$$

In other words, the entries in the n-th row of Pascal's triangle have a sum equal to 2^n. Here, we use for convenience the convention that the top entry in Pascal's triangle represents the 0-th row.

EXAMPLE 8.4.2. Suppose that in the expansion of $(1+x)^n$, the coefficient of x^2 is equal to 28. Clearly the term corresponding to x^2 is equal to

$$\binom{n}{2}x^2,$$

so that

$$\frac{n!}{2!(n-2)!}=\frac{n(n-1)}{2}=28.$$

Hence $n=8$. In fact, the first four terms of the expansion of $(1+x)^8$ sum to

$$\binom{8}{0}+\binom{8}{1}x+\binom{8}{2}x^2+\binom{8}{3}x^3=1+8x+28x^2+56x^3.$$

EXAMPLE 8.4.3. We have

$$\left(3x^2-\frac{1}{2x}\right)^9=\sum_{k=0}^{9}\binom{9}{k}(3x^2)^{9-k}\left(-\frac{1}{2x}\right)^k=\sum_{k=0}^{9}\binom{9}{k}(-1)^k\frac{3^{9-k}}{2^k}x^{18-3k}.$$

It follows that the term independent of x in this expansion is given by $k = 6$. This is equal to

$$\binom{9}{6}(-1)^6 \frac{3^3}{2^6} = \frac{567}{16}.$$

EXAMPLE 8.4.4. We have

$$\left(x + \frac{1}{x^2}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} x^{12-k} \left(\frac{1}{x^2}\right)^k = \sum_{k=0}^{12} \binom{12}{k} x^{12-3k}.$$

To obtain the coefficient of x^3, we take $k = 3$. This is given by $\binom{12}{3} = 220$. To obtain the coefficient of x^{-9}, we take $k = 7$. This is given by $\binom{12}{7} = 792$. To obtain the coefficient of the constant term, we take $k = 4$. This is given by $\binom{12}{4} = 495$.

8.5. Application to Probability Theory

We begin by studying a very simple example.

EXAMPLE 8.5.1. Suppose that two coins are tossed, and that each is either heads (h) or tails (t) equally likely. Then the following four outcomes are equally likely:

$$\begin{array}{cc} (h,h) & (h,t) \\ (t,h) & (t,t) \end{array}$$

- The probability of a pair of heads is $1/4$, representing one of the four possible outcomes.
- The probability of both coins landing the same way is $1/2$, representing two of the four possible outcomes.
- The probability of getting at least one head is $3/4$, representing three of the four possible outcomes.

In general, if there are several equally likely, mutually exclusive and collectively exhaustive outcomes of an experiment, then the probability of an event $\mathcal{E}$ is given by

$$p(\mathcal{E}) = \frac{\text{number of outcomes favourable to } \mathcal{E}}{\text{total number of outcomes}}.$$

EXAMPLE 8.5.2. Suppose that two dice are thrown. What is the probability that the sum of the two numbers is equal to 7? To answer this question, let us look at all the possible and equally likely outcomes:

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \underline{(1,6)} \\ (2,1) & (2,2) & (2,3) & (2,4) & \underline{(2,5)} & (2,6) \\ (3,1) & (3,2) & (3,3) & \underline{(3,4)} & (3,5) & (3,6) \\ (4,1) & (4,2) & \underline{(4,3)} & (4,4) & (4,5) & (4,6) \\ (5,1) & \underline{(5,2)} & (5,3) & (5,4) & (5,5) & (5,6) \\ \underline{(6,1)} & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array}$$

The cases where the sum of the two numbers shown is equal to 7 has been underlined. These represent 6 out of 36 equally likely outcomes. It follows that the probability is equal to $6/36 = 1/6$.

EXAMPLE 8.5.3. Let us return to Example 8.3.4 concerning drawing 5 cards out of a deck of 52 playing cards.

- The probability of drawing exactly two A's and three K's is equal to

$$\frac{C(4,2)\times C(4,3)}{C(52,5)}=\frac{1}{108290}.$$

- The probability of drawing 5 cards with a total value of 7 is equal to

$$\frac{C(4,4)\times C(4,1)+C(4,3)\times C(4,2)}{C(52,5)}=\frac{1}{92820}.$$

In probability theory, we often have to deal with multiple events. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two events.

- By the event $\mathcal{A}\cap\mathcal{B}$, we mean the event that both $\mathcal{A}$ and $\mathcal{B}$ occur.
- By the event $\mathcal{A}+\mathcal{B}$, we mean the event that at least one of $\mathcal{A}$ or $\mathcal{B}$ occurs.

These are related to events $\mathcal{A}$ and $\mathcal{B}$ in probability as follows.

PROBABILITY OF DOUBLE EVENTS. *For any two events $\mathcal{A}$ and $\mathcal{B}$, we have*

$$p(\mathcal{A}+\mathcal{B})=p(\mathcal{A})+p(\mathcal{B})-p(\mathcal{A}\cap\mathcal{B}).$$

An important result in probability theory concerns independent events. Roughly speaking, two events $\mathcal{A}$ and $\mathcal{B}$ are independent if each does not affect the outcome of the other.

PROBABILITY OF INDEPENDENT EVENTS. *For any two independent events $\mathcal{A}$ and $\mathcal{B}$, we have*

$$p(\mathcal{A}\cap\mathcal{B})=p(\mathcal{A})\times p(\mathcal{B}).$$

EXAMPLE 8.5.4. Suppose that 6 people out of 10 are to be chosen not in any particular order, and that A and B are two of these 10 people. Assume that each of the 10 people is equally likely to be chosen. Let $\mathcal{A}$ denote the event that A is chosen, and let $\mathcal{B}$ denote the event that B is chosen.

- What is the probability $p(\mathcal{A})$ that A is chosen? To answer this question, let us consider the number of ways of choosing 6 people out of 10 to include A. Clearly we choose A and then choose 5 from the remaining 9 people. Hence the number of ways of choosing 6 people out of 10 to include A is equal to $C(9,5)$. On the other hand, the number of ways of choosing 6 people out of 10 is equal to $C(10,6)$. It follows that the probability that A is chosen is also equal to

$$p(\mathcal{A})=\frac{C(9,5)}{C(10,6)}=\frac{3}{5}.$$

- Similarly, the probability that B is chosen is also equal to

$$p(\mathcal{B})=\frac{C(9,5)}{C(10,6)}=\frac{3}{5}.$$

- What is the probability $p(\mathcal{A}\cap\mathcal{B})$ that both A and B are chosen? To answer this question, let us consider the number of ways of choosing 6 people out of 10 to include both A and B. Clearly we choose both A and B and then choose 4 from the remaining 8 people. Hence the number of ways of choosing 6 people out of 10 to include both A and B is equal to $C(8,4)$. On the other hand, the number of ways of choosing 6 people out of 10 is equal to $C(10,6)$. It follows that the probability that both A and B are chosen is equal to

$$p(\mathcal{A}\cap\mathcal{B})=\frac{C(8,4)}{C(10,6)}=\frac{1}{3}.$$

- What is the probability $p(\mathcal{A}+\mathcal{B})$ that at least one of A or B is chosen? Let us consider instead the complementary event that neither A nor B is chosen. To do so, we simply choose all 6 people from the remaining 8, and the number of ways of doing this is equal to $C(8,6)$. It follows that the probability that neither A nor B is chosen is equal to

$$\frac{C(8,6)}{C(10,6)} = \frac{2}{15}.$$

Hence the probability that at least one of A or B is chosen is equal to

$$p(\mathcal{A}+\mathcal{B}) = \frac{13}{15}.$$

- Note that $p(\mathcal{A}+\mathcal{B}) = p(\mathcal{A}) + p(\mathcal{B}) - p(\mathcal{A}\cap\mathcal{B})$.
- Note that $p(\mathcal{A}\cap\mathcal{B}) \neq p(\mathcal{A}) \times p(\mathcal{B})$, so that the events $\mathcal{A}$ and $\mathcal{B}$ are not independent. It is clear that A not being chosen enhances the chances of B being chosen.

Problems for Chapter 8

1. Consider the collection $\mathcal{S}$ of all 7-digit numbers where each digit is non-zero.
   a) How many numbers are there in the collection $\mathcal{S}$?
   b) How many numbers in $\mathcal{S}$ have distinct digits?
   c) How many numbers in $\mathcal{S}$ have 1 as its first digit?
   d) How many numbers in $\mathcal{S}$ have distinct digits as well as 2 as its first digit and 4 as its last digit?

2. Consider the collection $\mathcal{S}$ of all 5-digit numbers where each digit is odd.
   a) How many numbers are there in the collection $\mathcal{S}$?
   b) How many numbers in $\mathcal{S}$ have distinct digits?
   c) How many numbers in $\mathcal{S}$ have 1 as its first digit?
   d) How many numbers in $\mathcal{S}$ have distinct digits as well as 1 as its first digit and 3 as its last digit?

3. Determine the number of ways of arranging the letters of *MACQUARIEUNIVERSITYSYDNEY* and leave your answer in terms of factorials.

4. Determine the number of ways of arranging the letters of *SYDNEYOLYMPICGAMES* and leave your answer in terms of factorials.

5. In how many ways can one choose 12 people from 20 people and seat them
   a) in a row from left to right?
   b) in a circle?
   c) in a square with 3 on each side?
   d) in a triangle with 4 on each side?
   e) in two rows of 6 facing each other?

6. In an examination paper, there are 8 questions on modern history and 7 questions on ancient history. A candidate is required to choose exactly 10 questions.
   a) In how many ways can this be achieved?
   b) In how many ways can this be achieved if the candidate is required additionally to choose exactly 5 questions in modern history?
   c) In how many ways can this be achieved if the candidate is required additionally to choose at least 5 questions in modern history?

7. There are 20 men and 20 women in a class. A mad professor decides to choose 10 students and give them extra assignments.
   a) In how many ways can the professor choose his victims?
   b) In how many ways can the professor choose his victims if he decides to choose exactly 5 men and 5 women?
   c) In how many ways can the professor choose his victims if he decides to choose at least 5 men?
   d) In how many ways can the professor choose his victims if he decides to choose at most 6 women?

8. We wish to choose 10 cards from a usual deck of 52 playing cards.
   a) In how many ways can be achieve this?
   b) In how many ways can we achieve this if we are required to choose all cards from the same suit?
   c) In how many ways can we achieve this if we are required to choose exactly 3 aces and 3 kings?
   d) In how many ways can we achieve this if we are required to choose cards of different values (assuming that the 13 cards in each suit are of different values)?

9. If we roll two dice, what is the probability that the sum of the two numbers is a multiple of 3?

10. Suppose that you are one of 10 candidates for election to a small committee of 3 people. Suppose further that each candidate is equally likely to be elected.
    a) What is the probability that you will be successful?
    b) Your best friend is also one of the candidates. What is the probability that both of you are successful?

11. We wish to elect 10 members to a committee from 100 candidates, and you are one of the candidates.
    a) What is the probability that you are elected?
    b) Two of your friends are also among the candidates.
       (i) What is the probability that you and both your friends are elected?
       (ii) What is the probability that you and exactly one of your two two friends are elected?
       (iii) What is the probability that you and at least one of your two friends are elected?
       (iv) What is the probability that both your friends are elected but you are not?

− ∗ − ∗ − ∗ − ∗ − ∗ −