Groups:
Their Presentations and Representations

COURSE NOTES FOR MATH337

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How does this differ from other introductions to group theory?

It is usual to introduce students to group theory via a course on groups, rings and fields rather than group theory in isolation. It is generally felt that students benefit from the parallel between the axiomatic development of groups and rings. Regrettably students do not see it that way and fail to be excited by a parallel between one unfamiliar theory and another.

These notes confine themselves to group theory with the aim of giving the student a wide experience there. By the time they learn about rings the parallel will be between the familiar and the unfamiliar.

But it might be objected that this will result in students learning more group theory than they need as an undergraduate. Certainly it would be a mistake to lead students into the technical depths of the subject, but there is a lot of room for breadth. What an undergraduate needs is a wide variety of mathematical experiences, and group theory can provide a lot of these.

The obvious experience is that of the development of an axiomatic system. Historically, group theory was the forerunner in this “modern” approach to mathematics. And because there are few axioms and a wide variety of different types of examples students appreciate the experience of axiomatic abstraction within group theory much better than they do in many other areas, such as the theory of vector spaces.

Instead of merely illustrating the definitions and theorems with a few well-chosen examples we have attempted to inundate the student with such a variety of examples that they come to value the theory as a way of organizing this wide experience.

But mathematics is in the process of making another turn and group theory is once again one of the leaders. Computation has moved from merely being a tool to being integrated into the theoretical development. Far from being a specialist area within group theory, computational group theory has become mainstream. And similar things are starting to happen in other areas of mathematics.

This has shifted the emphasis somewhat from the axioms to the description of a group through generators and relations. Of course only finitely presented groups are accessible in this way and often groups arise independently of any particular presentation. But many important groups that arise in applications do so as a presentation.

This justifies the inclusion of a chapter on the Todd-Coxeter algorithm. It is by no means a recent thing but it has become increasingly important. With the accompanying discussion of the Word Problem (stated but not proved here) it also has the merit of introducing the student to the idea of the non-computable within a purely mathematical context. (Somehow to a mathematician the unsolvability of the Word Problem seems to hit home more than the unsolvability of the Halting Problem despite the two being equivalent.)

Group presentations can be accepted quite readily at an intuitive level, which is the way they are first introduced and used. A later chapter develops the concept via free groups in a more rigorous way. Another chapter, on infinite abelian groups, provides a valuable contrast to this computational flavour and provides an excuse for talking about Zorn's Lemma.

Another topic, which is normally considered too advanced for an introductory course on group theory, is representation theory. This is because a completely self-contained account really needs a fair knowledge of rings and modules. However the deepest aspects can be isolated into a single Fundamental Theorem of Characters ("the irreducible character form an orthonormal basis for the space of class functions") which can be taken on faith just as the Fundamental Theorem of Algebra is when complex numbers are first encountered. From there everything can be built up with the help of a fair bit of standard linear algebra, thereby reinforcing the student's, often patchy, knowledge. The other benefit that has been found is that in constructing the character table for a group the student has to integrate a lot of elementary group theory. In particular the act of inducing characters from quotient groups in specific examples has been shown to cement the idea of a quotient group in a way that no amount of theory can.

These notes have been used quite effectively in conjunction with an individual project where each student receives the presentation of a group (a non-abelian group of order 16). The task is first to use the Todd-Coxeter algorithm to produce a group table and then, largely by examining the quotient structure and using the technique of inducing characters from quotient groups, construct the character table for their group. It has been found that because each student is working with their own presentation, and because so much structure seems to miraculously grow from such a tiny "seed" they are generally highly motivated and come away from the course with a much better grasp of the subject than with the traditional theorem-proof approach.

That is not to say that there are no proofs. Indeed everything used here is proved with the exception of the Fundamental Theorem of Characters and the Unsolvability of the Word Problem. It is just that the emphasis is a little more on doing than proving.

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