APPENDIX A: GROUPS OF SMALL ORDER

1

	1
1	1

2.1 $\langle A | A^2 \rangle$

	1	2
1	1	2
2	2	1

3.1 $\langle A | A^3 \rangle$

	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

4.1 = $\langle A | A^4 \rangle$

	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

4.2 = $\langle A, B | A^2, B^2, AB=BA \rangle$

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

5.1 $\langle A | A^5 \rangle$

	1	2	3	4	5
1	1	2	3	4	5
2	2	3	4	5	1
3	3	4	5	1	2
4	4	5	1	2	3
5	5	1	2	3	4

6.1 $\langle A | A^6 \rangle$

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	5	6	1
3	3	4	5	6	1	2
4	4	5	6	1	2	3
5	5	6	1	2	3	4
6	6	1	2	3	4	5

6.2 $\langle A, B | A^3, B^2, AB = BA^{-1} \rangle$

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	2	5	6	4
4	4	5	6	1	2	3
5	5	6	4	3	1	2
6	6	4	5	2	3	1

7.1 $\langle A | A^7 \rangle$

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1
3	3	4	5	6	7	1	2
4	4	5	6	7	1	2	3
5	5	6	7	1	2	3	4
6	6	7	1	2	3	4	5
7	7	1	2	3	4	5	6

8.1 $\langle A | A^8 \rangle$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	5	6	7	8	1
3	3	4	5	6	7	8	1	2
4	4	5	6	7	8	1	2	3
5	5	6	7	8	1	2	3	4
6	6	7	8	1	2	3	4	5
7	7	8	1	2	3	4	5	6
8	8	1	2	3	4	5	6	7

8.2 $\langle A, B | A^4, B^2, AB = BA \rangle$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	6	7	8	1	2	3	4
6	6	7	8	5	2	3	4	1
7	7	8	5	6	3	4	1	2
8	8	5	6	7	4	1	2	3

8.3 $\langle A, B, C | A^2, B^2, C^2, AB = BA, AC = CA, BC = CB \rangle$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

8.4 $\langle A, B | A^4, B^2, AB = BA^{-1} \rangle$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	8	5	6	7
3	3	4	1	2	7	8	5	6
4	4	1	2	3	6	7	8	5
5	5	6	7	8	1	2	3	4
6	6	7	8	5	4	1	2	3
7	7	8	5	6	3	4	1	2
8	8	5	6	7	2	3	4	1

8.5 $\langle A, B | A^4, B^2 = A^2, AB = BA^{-1} \rangle$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	8	5	6	7
3	3	4	1	2	7	8	5	6
4	4	1	2	3	6	7	8	5
5	5	6	7	8	3	4	1	2
6	6	7	8	5	2	3	4	1
7	7	8	5	6	1	2	3	4
8	8	5	6	7	4	1	2	3

9.1 ⟨A | A^9 ⟩

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	4	5	6	7	8	9	1
3	3	4	5	6	7	8	9	1	2
4	4	5	6	7	8	9	1	2	3
5	5	6	7	8	9	1	2	3	4
6	6	7	8	9	1	2	3	4	5
7	7	8	9	1	2	3	4	5	6
8	8	9	1	2	3	4	5	6	7
9	9	1	2	3	4	5	6	7	8

9.2 ⟨A, B | A^3, B^3, AB = BA⟩

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	1	5	6	4	8	9	7
3	3	1	2	6	4	5	9	7	8
4	4	5	6	7	8	9	1	2	3
5	5	6	4	8	9	7	2	3	1
6	6	4	5	9	7	8	3	1	2
7	7	8	9	1	2	3	4	5	6
8	8	9	7	2	3	1	5	6	4
9	9	7	8	3	1	2	6	4	5

10.1 ⟨A | A^{10} ⟩

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	1
3	3	4	5	6	7	8	9	10	1	2
4	4	5	6	7	8	9	10	1	2	3
5	5	6	7	8	9	10	1	2	3	4
6	6	7	8	9	10	1	2	3	4	5
7	7	8	9	10	1	2	3	4	5	6
8	8	9	10	1	2	3	4	5	6	7
9	9	10	1	2	3	4	5	6	7	8
10	10	1	2	3	4	5	6	7	8	9

10.2 ⟨A, B | A^5, B^2, AB = BA^{-1} ⟩

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	1	10	6	7	8	9
3	3	4	5	1	2	9	10	6	7	8
4	4	5	1	2	3	8	9	10	6	7
5	5	1	2	3	4	7	8	9	10	6
6	6	7	8	9	10	1	2	3	4	5
7	7	8	9	10	6	5	1	2	3	4
8	8	9	10	6	7	4	5	1	2	3
9	9	10	6	7	8	3	4	5	1	2
10	10	6	7	8	9	2	3	4	5	1

11.1 ⟨A | A^{11} ⟩

	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	1
3	3	4	5	6	7	8	9	10	11	1	2
4	4	5	6	7	8	9	10	11	1	2	3
5	5	6	7	8	9	10	11	1	2	3	4
6	6	7	8	9	10	11	1	2	3	4	5
7	7	8	9	10	11	1	2	3	4	5	6
8	8	9	10	11	1	2	3	4	5	6	7
9	9	10	11	1	2	3	4	5	6	7	8
10	10	11	1	2	3	4	5	6	7	8	9
11	11	1	2	3	4	5	6	7	8	9	10

12.1 $\langle A \mid A^{12} \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	3	4	5	6	7	8	9	10	11	12	1
3	3	4	5	6	7	8	9	10	11	12	1	2
4	4	5	6	7	8	9	10	11	12	1	2	3
5	5	6	7	8	9	10	11	12	1	2	3	4
6	6	7	8	9	10	11	12	1	2	3	4	5
7	7	8	9	10	11	12	1	2	3	4	5	6
8	8	9	10	11	12	1	2	3	4	5	6	7
9	9	10	11	12	1	2	3	4	5	6	7	8
10	10	11	12	1	2	3	4	5	6	7	8	9
11	11	12	1	2	3	4	5	6	7	8	9	10
12	12	1	2	3	4	5	6	7	8	9	10	11

12.2 $\langle A, B \mid A^6, B^2, AB = BA \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	3	4	5	6	1	8	9	10	11	12	7
3	3	4	5	6	1	2	9	10	11	12	7	8
4	4	5	6	1	2	3	10	11	12	7	8	9
5	5	6	1	2	3	4	11	12	7	8	9	10
6	6	1	2	3	4	5	12	7	8	9	10	11
7	7	8	9	10	11	12	1	2	3	4	5	6
8	8	9	10	11	12	7	2	3	4	5	6	1
9	9	10	11	12	7	8	3	4	5	6	1	2
10	10	11	12	7	8	9	4	5	6	1	2	3
11	11	12	7	8	9	10	5	6	1	2	3	4
12	12	7	8	9	10	11	6	1	2	3	4	5

12.3 $\langle A, B \mid A^6, B^2, AB = BA^{-1} \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	3	4	5	6	1	12	7	8	9	10	11
3	3	4	5	6	1	2	11	12	7	8	9	10
4	4	5	6	1	2	3	10	11	12	7	8	9
5	5	6	1	2	3	4	9	10	11	12	7	8
6	6	1	2	3	4	5	8	9	10	11	12	7
7	7	8	9	10	11	12	1	2	3	4	5	6
8	8	9	10	11	12	7	6	1	2	3	4	5
9	9	10	11	12	7	8	5	6	1	2	3	4
10	10	11	12	7	8	9	4	5	6	1	2	3
11	11	12	7	8	9	10	3	4	5	6	1	2
12	12	7	8	9	10	11	2	3	4	5	6	1

12.4 $\langle A, B \mid A^3, B^4, AB = BA^{-1} \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	3	1	6	4	5	8	9	7	12	10	11
3	3	1	2	5	6	4	9	7	8	11	12	10
4	4	5	6	7	8	9	10	11	12	1	2	3
5	5	6	4	9	7	8	11	12	10	3	1	2
6	6	4	5	8	9	7	12	10	11	2	3	1
7	7	8	9	10	11	12	1	2	3	4	5	6
8	8	9	7	12	10	11	2	3	1	6	4	5
9	9	7	8	11	12	10	3	1	2	5	6	4
10	10	11	12	1	2	3	4	5	6	7	8	9
11	11	12	10	3	1	2	5	6	4	9	7	8
12	12	10	11	2	3	1	6	4	5	8	9	7

12.5 $\langle A, B, C \mid A^2, B^2, C^2, AB{=}BA, AC{=}CB, BC{=}CBA \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	3	7	8	5	6	12	11	10	9
3	3	4	1	2	8	7	6	5	10	9	12	11
4	4	3	2	1	6	5	8	7	11	12	9	10
5	5	6	7	8	9	10	11	12	1	2	3	4
6	6	5	8	7	11	12	9	10	4	3	2	1
7	7	8	5	6	12	11	10	9	2	1	4	3
8	8	7	6	5	10	9	12	11	3	4	1	2
9	9	10	11	12	1	2	3	4	5	6	7	8
10	10	9	12	11	3	4	1	2	8	7	6	5
11	11	12	9	10	4	3	2	1	6	5	8	7
12	12	11	10	9	2	1	4	3	7	8	5	6

NUMBER OF GROUPS OF EACH ORDER UP TO 60

1	1	**6**	2	**11**	1	**16**	15	**21**	2	**26**	2	**31**	1	**36**	14	**41**	1	**46**	2	**51**	1	56	**13**
2	1	**7**	1	**12**	5	**17**	1	**22**	2	**27**	5	**32**	51	**37**	1	**42**	6	**47**	1	**52**	5	57	**2**
3	1	**8**	5	**13**	1	**18**	5	**23**	1	**28**	4	**33**	1	**38**	2	**43**	1	**48**	52	**53**	1	58	**2**
4	2	**9**	2	**14**	2	**19**	1	**24**	15	**29**	1	**34**	2	**39**	2	**44**	4	**49**	2	**54**	15	59	**1**
5	1	**10**	2	**15**	1	**20**	5	**25**	2	**30**	4	**35**	1	**40**	14	**45**	2	**50**	5	**55**	2	60	**13**

APPENDIX B: CHARACTER TABLE SUMMARY

conjugacy classes	Γ_1	Γ_2	.....	Γ_s	Γ_{s+1}	.....	Γ_m
size	1	h_2	.....	h_s	h_{s+1}	.....	h_n
χ_1	1	1	.....	1	1	.....	1
χ_2	1			roots of unity			
......	.....						
χ_r	1						
χ_{r+1}	n_{r+1}			a sum of roots of unity			
.....	.....						
χ_m	n_m						
order	1	k_2	.....	k_s	k_{s+1}	.....	k_n

(1) m = # conjugacy classes
[= # irreducible characters]
	(2) $n_i = \deg \chi_i$

(3) $r = \|G/G'\|$ [= # linear characters]	(4) $s = \|Z(G)\|$ [= # classes of size 1]
(5) $\Sigma n_i^2 = \|G\|$ [follows from column orthonormality]	(6) $\Sigma h_i = \|G\|$ [class equation]
(7) (Column orthonormality) $\sum_i \chi_i(\Gamma_j)\overline{\chi_i(\Gamma_k)} = 0$ if $j \neq k$; $\sum_i \left\|\chi_i(\Gamma_j)\right\|^2 = \frac{\|G\|}{h_j}$	(8) (Row orthonormality) $\sum_j \chi_i(\Gamma_j)\overline{\chi_k(\Gamma_j)} = 0$ if $i \neq k$; $\sum_j \left\|\chi_i(\Gamma_j)\right\|^2 = \|G\|$
(9) $\chi_i(\Gamma_j^{-1}) = \overline{\chi_i(\Gamma_j)}$ [columns corresponding to inverses are conjugate].	(10) The conjugate of an irreducible character is an irreducible character.
(11) If $\Gamma_j = \Gamma_j^{-1}$ then each $\chi_i(\Gamma_j)$ is real.	(12) The linear characters form a group, under multiplication, isomorphic to $\|G/H\|$.
(13) ker ρ_i (where ρ_i is a representation for which χ_i is the character) is the union of all conjugacy classes Γ for which $\chi_i(\Gamma) = n_i$.	(14) Every normal subgroup is one of these kernels or the intersection of two or more of these kernels.
(15) Every irreducible character χ of G/H induces an irreducible character of G. [Each conjugacy class Γ in G/H corresponds to one or more classes of G. These classes map to $\chi(\Gamma)$.]	(16) $\chi_i(\Gamma_j)$ is a sum of n_i k_j'th roots of unity. [In particular, if f $k_i = 2$ then $\chi_i(\Gamma_j)$ is: ± 1 if $n_i = 1$; 0 or ± 2 if $n_i = 2$; ± 1 or ± 3 if $n_i = 3$.]
(17) If π is a permutation character $\pi(\Gamma_j)$ is the number of symbols fixed by each element of Γ_j. [It is usually reducible]	(18) If $\chi = \Sigma c_i \chi_i$ is the decomposition into irreducibles then $\langle \chi \mid \chi \rangle = \frac{1}{\|G\|}\sum_j \left\|\chi(\Gamma_j)\right\|^2 = \Sigma c_i^2$. and $\langle \chi \mid \chi_i \rangle = \frac{1}{\|G\|}\sum_j \chi(\Gamma_j)\overline{\chi_i(\Gamma_j)} = c_i$. [In particular if $\langle \chi \mid \chi \rangle = 1$, χ is irreducible and if $\langle \chi \mid \chi \rangle = 2$, χ is the sum of two distinct irreducible characters.]

APPENDIX C: A GROUP OF ORDER 12

$G = \langle A,B,C \mid A^3, B^2, C^6, C^4, AB^{-1}AB, AC = CA, BC = CB \rangle$

By the Todd-Coxeter algorithm we can show that G has order 12 and we can obtain its group table as follows:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	1	3	5	8	10	12	7	9	11
3	3	5	1	6	2	4	9	11	7	12	8	10
4	4	1	5	2	6	3	10	7	11	8	12	9
5	5	6	4	3	1	2	11	12	10	9	7	8
6	6	3	2	5	4	1	12	9	8	11	10	7
7	7	8	9	10	11	12	1	2	3	4	5	6
8	8	10	12	7	9	11	2	4	6	1	3	5
9	9	11	7	12	8	10	3	5	1	6	2	4
10	10	7	11	8	12	9	4	1	5	2	6	3
11	11	12	10	9	7	8	5	6	4	3	1	2
12	12	9	8	11	10	7	6	3	2	5	4	1

Inverses:

1	2 4	3	5	6	7	8 10	9	11	12

Orders of Elements:

elt	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
order	1	3	2	3	2	2	2	6	2	6	2	2

Order Profile:

order	1	2	3	6
number	1	7	2	2

Conjugacy Classes:

	C_1	C_2	C_3	C_4	C_5	C_6
	1	2 4	3 5 6	7	8 10	9 11 12
order	1	3	2	2	6	2

Class Equation: 12 = 1 + 1 + 2 + 2 + 3 + 3

Subgroups:
The proper non-trivial subgroups are:

G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}
1 3	1 5	1 6	1 7	1 9	1 11	1 12	1 2 4	1 3 7 9	1 5 7 11	1 6 7 12

G_{12}	G_{13}	G_{14}
1 2 4 7 8 10	1 2 3 4 5 6	1 2 4 9 11 12

normal subgroups are shaded

The reasoning which ensures we have them all is as follows:

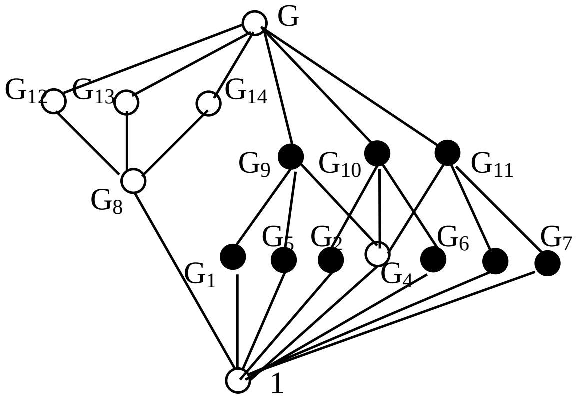
Order 1: {1}
Order 2: These must be cyclic, generated by an element of order 2, so the only possibilities are: G_1 - G_7.
Order 3: These must be cyclic, generated by an element of order 3, so the only possibility is G_8.
Order 4: As there are no elements of order 4 there are no cyclic subgroups of order 4. The subgroups of order 4 must therefor be isomorphic to V_4 and so be generated by two commuting elements of order 2. These subgroups are thus G_9, G_{10}, G_{11}.
Order 6: A cyclic subgroup of order 6 is generated by an element of order 6 and so the only one is G_{12}.
A non-cyclic subgroup of order 6 has index 2 and so must be normal. It must therefore consist of entire conjugacy classes, two of which must be C_1, C_2. The other class must be C_3 or C_6, giving G_{13}, G_{14}.
Order 12: This is G itself.
Subgroups are normal if and only if they are unions of conjugacy classes so of the subgroups G_1 - G_{14} the normal subgroups are G_4, G_8, G_{12}, G_{13}, G_{14}.

Centralisers:

elt	centraliser	subgp	#conjugates
1	G	G	1
2	1 2 4 7 8 10	G_{12}	2
3	1 3 7 9	G_9	3
4	1 2 4 7 8 10	G_{12}	2
5	1 5 7 11	G_{10}	3
6	1 6 7 12	G_{11}	3
7	G	G	1
8	1 2 4 7 8 10	G_{12}	2
9	1 3 7 9	G_9	3
10	1 2 4 7 8 10	G_{12}	2
11	1 5 7 11	G_{10}	3
12	1 6 7 12	G_{11}	3

Lattice of Subgroups:

● denotes a non-normal subgroup ○ denotes a normal subgroup



Cosets:

subgp	LEFT COSETS						RIGHT COSETS					
G_1	1 3	2 6	4 5	7 9	8 12	10 11	1 3	2 5	4 6	7 9	8 11	10 12
G_2	1 5	2 3	4 6	7 11	8 9	10 12	1 5	2 6	3 4	7 11	8 12	9 10
G_3	1 6	2 3	4 5	7 12	8 11	9 10	1 6	2 3	4 5	7 12	8 9	10 11
G_4	1 7	2 8	3 9	4 10	5 11	6 12	1 7	2 8	3 9	4 10	5 11	6 12
G_5	1 9	2 12	3 7	4 11	5 10	6 8	1 9	2 11	3 7	4 12	5 8	6 10
G_6	1 11	2 9	3 8	4 12	5 7	6 10	1 11	2 12	3 10	4 9	5 7	6 8
G_7	1 12	2 11	3 10	4 9	5 8	6 7	1 12	2 9	3 8	4 11	5 10	6 7
G_8	1 2 4	3 5 6	7 8 10	9 11 12			1 2 4	3 5 6	7 8 10	9 11 12		
G_9	1 3 7 9	2 6 8 12	4 5 10 11				1 3 7 9	2 5 8 11	4 6 10 12			
G_{10}	1 5 7 11	2 3 8 9	4 6 10 12				1 5 7 11	2 6 8 12	3 4 9 10			
G_{11}	1 6 7 12	2 5 8 11	3 4 9 10				1 6 7 12	2 3 8 9	4 5 10 11			
G_{12}	1 2 4 7 8 10	3 5 6 9 12					1 2 4 7 8 10	3 5 6 9 12				
G_{13}	1 2 3 4 5 6	7 8 9 10 11 12					1 2 3 4 5 6	7 8 9 10 11 12				
G_{14}	1 2 4 9 11 12	3 5 6 7 8 10					1 2 4 9 11 12	3 5 6 7 8 10				

Quotient Groups: (representing cosets by representatives)

G/G_4:

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	5	1	6	2	4
4	4	1	5	2	6	3
5	5	6	4	3	1	2
6	6	3	2	5	4	1

G/G_8:

	1	**3**	**7**	**9**
1	1	3	7	9
3	3	1	9	7
7	7	9	1	3
9	9	7	3	1

G/G_{12}:

	1	**3**
1	1	3
3	3	1

G/G_{13}:

	1	**7**
1	1	7
7	7	1

G/G_{14}:

	1	**3**
1	1	3
3	3	1

WARNING: In all these cases the group table for the quotient could be found directly from the group table for G. This will not always be the case. If the product of two representatives does not happen to be a representative, one should replace the product by the representative of the coset in which that product occurs.

Character Table

	C_1	C_2	C_3	C_4	C_5	C_6
order	**1**	**3**	**2**	**2**	**6**	**2**
size	**1**	**2**	**3**	**1**	**2**	**3**
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	1	1	1	–1	–1	–1
χ_5	1	1	–1	–1	–1	1
χ_6	2	–1	0	–2	1	0
orders	**1**	**3**	**2**	**2**	**6**	**2**

Explanation

The conjugacy classes of $H = G/G_4$ are {1}, {2, 4}, {3, 5, 6}.
It has a normal subgroup K = {1,2,4}. H/K is a cyclic group of order 2 with character table:

order	**1**	**2**
size	**1**	**1**
χ_1	1	1
χ_2	1	–1
orders	**1**	**2**

The character table for H is thus:

class	1	2	3	these are representatives from each class
size	1	2	3	
χ_1	1	1	1	trivial representation
χ_2	1	1	–1	inducing from H/K
χ_3	2	–1	0	by orthogonality
orders	1	3	2	

Hence the character table for G is:

	C_1	C_2	C_3	C_4	C_5	C_6
order	1	3	2	2	6	2
size	1	2	3	1	2	3
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4						
χ_5						
χ_6						

The remaining degrees n_4, n_5, n_6 must satisfy $n_4^2 + n_5^2 + n_6^2 = 6$ so we may take $n_4 = n_5 = 1$ and $n_6 = 2$.
Since each class is its own inverse the entries in the character table are all real.
Since the elements of C_3 and C_6 have order 2, the entries in those columns must be ±1 for the degree 1 representations and ±2 or 0 for the degree 2.
Also, the entries χ_{42} and χ_{52} are real cube roots of unity and so must be both 1.

Thus we may complete these columns as follows:

	C_1	C_2	C_3	C_4	C_5	C_6
size	1	2	3	1	2	3
χ_1	1	1	1	1	1	1
χ_2	1	1	–1	1	1	–1
χ_3	2	–1	0	2	–1	0
χ_4	1	1	1	a	b	–1
χ_5	1	1	–1	c	d	1
χ_6	2	x	0	e	f	0
orders	1	3	2	2	6	2

NOTE: The possibilities 1, 1, –1 for the last three entries in the 3rd and 6th columns violate the condition on the sum of squares of the entries down these columns.

By orthogonality with the first column we get $x = -1$. By orthogonality with the first and third rows we get $1 + 2 + 3 + a + 2b - 3 = 0$ whence $a + 2b = -3$, and $2 - 2 + 2a - 2b = 0$ whence $a = b$. Hence $a = b = -1$.
By orthogonality with the first and third rows we get $1 + 2 - 3 + c + 2d + 3 = 0$ whence $c + 2d = -3$, and $2 - 2 + 2c - 2d = 0$ whence $c = d$. Hence $c = d = -1$.
Finally by orthogonality with the first column, $e = -2$ and $f = 1$.

APPENDIX D: PRESENTATIONS OF GROUPS OF ORDER 16

Each of the following is a presentation of one of the 9 non-abelian groups of order 16.

1	$\langle A,B \mid A^4, B^4, (AB)^2, (A–1B)^2\rangle$
2	$\langle A,B \mid A^8, B^2, AB = BA^3\rangle$
3	$\langle A,B,C \mid A^4, B^2, C^2, AB = BA^{-1}, AC = CA, BC = CB\rangle$
4	$\langle A,B,C \mid A^2, B^4, C^2 = B^2, BCB = C, AB = BA, AC = CA\rangle$
5	$\langle A,B \mid A^4, B^2, (A^{-1}B)^4, (A^2B)^2\rangle$
6	$\langle A,B \mid A^2, B^8, AB^3 = BA\rangle$
7	$\langle A,B,C \mid A^2, B^2, C^2, A^{-1}BCA = BC, CAB = BCA\rangle$
8	$\langle A,B,C \mid A^2 = C^2, B^2, C^4, CAC = A, AB = BA, BC = CB\rangle$
9	$\langle A,B \mid A^4, B^4, AB = BA^{-1}\rangle$
10	$\langle A,B \mid A^8, A^4 = B^2, AB = BA^{-1}\rangle$
11	$\langle A,B \mid A^8, B^2, AB = BA^{-1}\rangle$
12	$\langle A,B \mid A^8, B^2, AB = BA^{-3}\rangle$
13	$\langle A,B,C \mid A^4, B^2 = A^2, C^2, AB = BA^{-1}, AC = CA, BC = CB\rangle$
14	$\langle A,B \mid A^4, B^4, BAB = A\rangle$
15	$\langle A,B \mid A^4, B^4, BA = AB^3\rangle$
16	$\langle A,B \mid A^2, B^8, A^{-1}BAB^{-1} = B^4\rangle$
17	$\langle A,B \mid A^8, B^2, (AB)^2 = B^2\rangle$
18	$\langle A,B,C \mid A^2, B^2, C^2, (ABC)^2 = (AB)^2, ABC = BCA\rangle$
19	$\langle A,B \mid A^2 = B^4, B^8, AB^{-1} = BA\rangle$
20	$\langle A,B,C \mid A^2, B^2, C^2, ABC = BCA = CAB\rangle$
21	$\langle A,B \mid A^8, B^2, BA^3B = A\rangle$
22	$\langle A,B,C \mid A^2, B^4, C^2, BCB = C, AB = BA, AC = CA\rangle$
23	$\langle A,B \mid A^4, B^4, (AB)^2, (AB^{-1})^2\rangle$
24	$\langle A,B \mid A^2, B^8, AB^3A = B\rangle$
25	$\langle A,B \mid A^4, B^4, (AB)^2, ABA^2B^{-1}A\rangle$
26	$\langle A,B \mid A^8, A^4 = B^2, BAB = A^3\rangle$
27	$\langle A,B \mid A^8, B^2, (A^{-1}B)^2\rangle$
28	$\langle A,B \mid A^2, B^8, B^{-1}A^{-1}BA = B^2\rangle$
29	$\langle A,B,C \mid A^4, B^2 = A^2, C^2, B^{-1}AB = A^3, AC = CA, BC = CB\rangle$
30	$\langle A,B \mid A^8, B^2, AB = BA^5\rangle$
31	$\langle A,B \mid A^4, B^4, (AB)^2 = B^2\rangle$
32	$\langle A,B,C \mid A^2, B^2, C^4, CAC = A, AB = BA, BC = CB\rangle$
33	$\langle A,B \mid A^8, B^2, (AB)^2\rangle$
34	$\langle A,B,C \mid A^2, B^2, C^2, ACB = CBA = BAC\rangle$
35	$\langle A,B,C \mid A^4, B^2, C^2, B^{-1}AB = A^3, AC = CA, BC = CB\rangle$
36	$\langle A,B \mid A^8, A^4 = B^2, A^4 = (AB)^2\rangle$