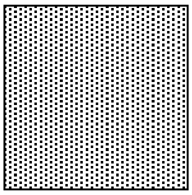
21. TOPOLOGICAL SPACES

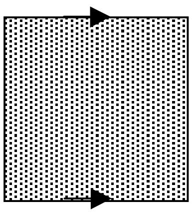
21.1 Some Surfaces, Familiar and Unfamiliar

Topology is the study of continuity. We want to call two geometric figures (such as curves or surfaces) "topologically equivalent" (the technical term is "homeomrphic") when one can be obtained from the other by a continuous transformation with a continuous inverse. Informally, these transformations are obtained by stretching the figures as though they were made of rubber; hence the term "rubber sheet geometry" for "topology". When the rubber is torn or glued the continuity is destroyed and topologically inequivalent figures are created.

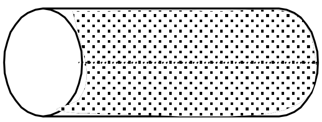
We will have a lot of fun cutting and gluing (as well as stretching) polygonal regions in the plane. For now let us consider a square.



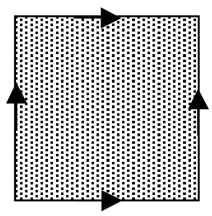
Consider gluing together some of the boundary points. To indicate that we will glue corresponding points on the two opposite sides, we place arrows thus:



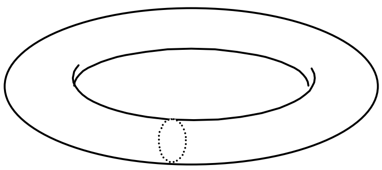
The effect of this is to obtain a **cylinder** where the dotted line is where the gluing occurred.



We can also contemplate gluing both pairs of opposite sides in the same way.

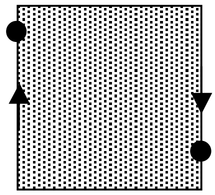


We can do this in two steps, first forming the cylinder as before, then wrap the open ends of the cylinder around to obtain a **torus** (or "doughnut").

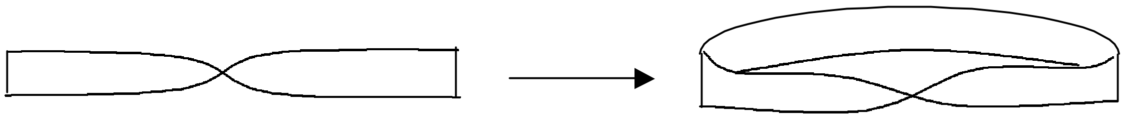


The dotted circle denotes the new glue.

The cylinder and torus are fairly familiar to us from elsewhere in mathematics. We are now going to obtain some stranger figures from our square, although the first one you may have met in preschool or kindergarten. We again glue one pair of opposite sides, but this time a point on the left edge is glued to the point on the right edge whose distance from the top end is the same as the distance from the bottom end of the other point.

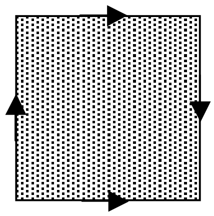


We think of this physically as stretching the whole figure in the horizontal direction to make a ribbon, giving it a vertical twist, and now bending and gluing the left and right edges. The result is called a Möbius Band.

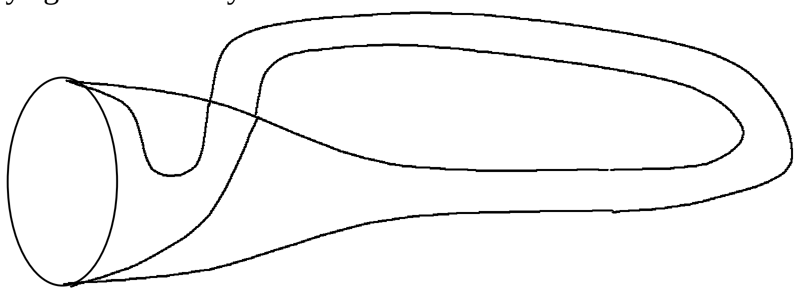


There are many interesting properties of the Möbius Band. In particular, it is **one-sided**. In integration over surfaces we meet the concept of orientability: the Möbius Band is not orientable.

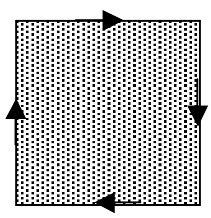
This leads us to contemplate the following.



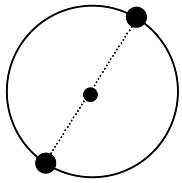
The result here is called a **Klein Bottle** but you will not have seen one lying around because it cannot exist in ordinary 3-dimensional Euclidean space $\mathbf{R}^3$. We can try to contruct it by first forming the cylinder by gluing the horizontal sides but then we need to bring the right-hand end up to the left-hand end from the inside: this cannot be done in $\mathbf{R}^3$ without tearing through the side of the cylinder, destroying the continuity.



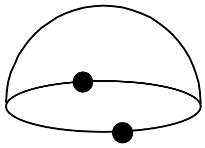
However the Klein Bottle does exist in R^4. In order to understand this we will need to make the notion of gluing and tearing more rigorous. Before we do that we look at the other case that we have so far omitted.



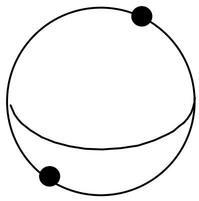
Rounding the corners of the square we see that this is topologically equivalent to a disc with antipodal points glued.



Now push the disc out of the plane to make the surface of a hemisphere with antipodal points on the boundary circle glued:



But we can also think of the points on the surface of the hemisphere that are not on the boundary as being glued with the point antipodally on the other (unseen) hemisphere. So we can think of our figure as the surface of a sphere with opposite (antipodal) points glued.



In fact we can think of this space (or figure) as being the collection of all lines through the origin in R^3. If you really want to see these lines as "points" you need to glue them all together along each line. Again this cannot be done continuously in R^3. The object we have formed is called the projective plane and it does not exist in R^3. However it does exist in R^4.

We can, however, contemplate it quite happily with a 3-dimensional model where the "points" are actually lines through the origin.

This shows how easily we are led into models for geometric objects where the "points" are not necessarily best viewed as points in Euclidean space. You have seen this before in algebra and calculus. Vector spaces are designed to have the properties of $\mathbf{R}^n$ where the elements are really points. Then we see that there are vector spaces whose elements are functions, or sequences and so on. We can include them because we define a vector space as an abstract set with extra structure: addition and scalar multiplication with certain axioms.

21.2 Frechet Spaces

Here we are interested in topological spaces. These are very general mathematical structures that allow us to speak of continuous functions between them.

Actually, vector spaces were designed so that we could speak of linear functions between them: linear functions preserve linear combinations so we needed to be able to speak of linear combinations in the vector spaces. Addition and scalar multiplicatyion allow us to do that.

You may have met the fact that a function $f : \mathbf{R} \to \mathbf{R}$ is continuous if and only if for every $x \in \mathbf{R}$ and every sequence (s_n) with limit x, the sequence $(f(s_n))$ has limit f(x). This is expressed more crudely as

$$\lim_{n\to\infty} f(s_n) = f\left(\lim_{n\to\infty} s_n\right).$$

This suggested to Frechet that we should define spaces where the concept of a sequence exists. Then we could define continuity as above.

We will adopt this approach even though it is no longer the accepted definition of topological space. Since we are not going to need uncountable cartesian products of spaces or functions, this will suffice.

A sequence in a set X is a function $s: \mathbf{N} \to X$ whose domain is the set **N** of natural numbers 0, 1, 2, … We write s_n for $s(n)$, and sometimes we write (s_n) or $s_0, s_1, s_2, \ldots$ for s. We call s_n the **n-th term** of s.

The sequence s in X is said to have a certain property **eventually** when there exists a natural number k such that the sequence (s_{k+n}), that is the sequence $s_k, s_{k+1}, s_{k+2}, \ldots$, has the property.

For example we say that s is eventually in a subset K of X when all but a finite number of terms of s are in K.

A sequence $h : \mathbf{N} \to \mathbf{N}$ is called **strictly increasing** when $h_0 < h_1 < h_2 < \ldots$

A sequence $t: \mathbf{N} \to X$ is called a **subsequence** of a sequence $s: \mathbf{N} \to X$ when there is a strictly increasing sequence $h : \mathbf{N} \to \mathbf{N}$ such that $t = s \circ h$.

If $f : X \to A$ is any function, each sequence $s: N \to X$ in X yields a sequence $f \circ s$. in A.

We sometimes denote $f \circ s$ by $f(s)$ or $f(s_0), f(s_1), f(s_2), \ldots$

Each subset K of X gives a subset $fK = \{f(x) \mid x \in K\}$ of A.

Without extra information on our set X we cannot sensibly discuss limits for sequences. What we do is to suppose that we are told which points of X are limits of which sequences. (Note that we call elements of X "points" although they may be nothing like points that you ar e familiar with.)

We also do not wish to assume that sequences have unique limits when they have one at all. A sequence could have no limits, one limit, two limits or even infinitely many limits.

For each $x \in X$, there is the **constant sequence** x, x, x, … where all terms are x.

Now we are ready for our general definition of a Frechet space.

Definition: A (**Frechet**) **space** is a set X together with a rule that assigns a set of elements $L(s) \subseteq X$ to each sequence s in X. The elements of L(s) are called limits of s. The following axioms are required to hold:

(1) if s is the constant sequence at x then $x \in L(s)$.;
(2) if sequences s and t are eventually equal then $L(s) = L(t)$;
(3) if t is a subsequence of s then $L(t) \subseteq L(s)$;
(4) if $x \notin L(s)$ then there is a subsequence t of s such that no subsequence of t has limit x.

For those who are happy with set theory we can think of the assignment of a set of limits to each sequence as a function:

$L : X^{\mathbf{N}} \to \wp X$

where $X^{\mathbf{N}}$ is the set of sequences in X and $\wp X$ is the set of subsets of X (called the **power set** of X).

Definition: A space X is called **Hausdorff** when each sequence has at most one limit (in other words a sequence cannot have two distinct limits).

Definition: A space X is called **compact** when each sequence s has a subsequence t that has at least one element.

21.3 Examples of Frechet Spaces

(1) Discrete Spaces

For any set X, put

$L(s) = \{x \in X \mid s \text{ is eventually constant at } x\}$.

In other words, the only sequences that have limits are the eventually constant sequences and the limit of such a sequence is the constant.

This is called a **discrete space**. Clearly it is Hausdorff. It isa compact if and only if X is finite.

(2) Chaotic (or indiscdrete) spaces

For any set X put L(s) = X for all sequences s. So every sequence has every point as a limit: chaotic!. If X has more than one point it is not Hausdorff. However it is compact.

(3) Metric spaces

A **premetric space** is a set X with a function $d : X \times X \to \mathbf{R}$ (called the **metric**) satisfying:

(i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, x) = 0$ for all $x \in X$;

(b) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$;

(c) $d(x, y) = d(y, x)$.

X is called a **metric space** when, in addition:

(d) $d(x, y) = 0$ implies that $x = y$.

For each $x \in X$ and positive number r, the **open ball with centre x and radius r** is defined by $B_r(x) = \{y \in X \mid d(x, y) < r\}$.

Then a sequence space is defined on the premetric space X by

$L(s) = \{x \in X \mid s \text{ is eventually in every open ball with centre } x\}$.

This is Hausdorff if and only if X is a metric space. Notice that every subset of a (pre)metric space becomes a (pre)metric space by simply restricting the metric.

(4) Euclidean space

Euclidean space of dimension n is:

$\mathbf{R}^n = \{x = (x_1, \ldots, x_n) \mid x_i \in R\}$.

It becomes a metric space with

$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots (x_n - y_n)^2}$.

So we obtain a sequence space as in example (3). It is Hausdorff but not compact.

(5) Projective space

Projective space of dimension n is $P_n(\mathbf{R}) = \{L \mid L \text{ is a line through the origin in } \mathbf{R}^{n+1}\}$.

A line L is a limit of a sequence $s_0, s_1, s_2, \ldots$of lines when there exist non -zero elements $s_0, s_1, s_2, \ldots$and x on the lines $S_0, S_1, S_2, \ldots$and L respectively such that x is a limit of the sequence $s_0, s_1, s_2, \ldots$in $\mathbf{R}^{n+1}$. We have met the projective plane $P_2(\mathbf{R})$ before.

(6) The Alexandrov Compactification of **N**

This is an example of a Frechet space that can be thought of as a "free -living convergent sequence". The set of points is $\hat{\mathbf{N}} = \{0, 1, 2, 3, \ldots, \infty\} = \mathbf{N} \cup \{\infty\}$.

A sequence s in $\hat{\mathbf{N}}$ is defined to have limit ∞ when, for each $k \in \mathbf{N}$, s is eventually bigger than k. A sequence s in $\hat{\mathbf{N}}$ has limit r $(< \infty)$ when it is eventually constant at r. This defines a compact Hausdoeff Fréchet space.

21.4 Open and Closed Subsets of a Frechet Space

We now look at some concepts associated with a space X.

A subset F of X is **closed** when, for all sequences s in F, every limit of s is in F. A subset U of X is **open** when its complement $X - U = \{x \in X \mid x \notin U\}$ is closed. Beware that subsets can be both open and closed and they can be neither open nor closed.

A neighbourhood (nhd) of a point $x \in X$ is a subset N of X such that every sequence s with limit x is eventually in N. In particular, x is N since the constant sequence x, x, , x, …has limit x. Clearly, if N is a nhd of x and $N \subseteq A$ then A is a nhd of x.

Proposition: A subset is open if and only if it is a neighbourhood of each of its points.
Proof: Assume U is open. Take $x \in U$ and let s be a sequence with limit x. Suppose s is not eventually in U. Then there is a subsequence t of s with t in $X - U$. By an axiom, x is a limit of t. But $U - X$ is closed so $x \notin U$, a contrradiction. So U is a nhd of x.

Assume U is not open. Then there exists a sequence s in $X - U$ with limit $x \in U$. Certainly s is not ever in U. So U is not a nhd of x.

Proposition: (a) $\varnothing$ and X are closed.
(b) If F_1 and F_2 are closed then so is $F_1 \cup F_2$.
(c) If F_i is closed for all $i \in I$ (any set) then the intersection $\bigcap_{i \in I} F_i = \{x \in X \mid x \in F_i \text{ for all } i \in I\}$ is closed.
Proof: (a) This is obvious.
(b) Suppose s is a sequence in $F_1 \cup F_2$ with limit $x \in X$. Either F_1 or F_2 must contain infinitely many terms of s; say F_1. Then we have a subsequence t of s in F_1. Since F_1 is closed and x is a limit of t, we have $x \in F_1 \subseteq F_1 \cup F_2$. So $F_1 \cup F_2$ is closed.
(c) Let s be a sequence in $\bigcap_{i \in I} F_i$ with limit x. Then s is in F_i and so $x \in F_i$ since F_i is closed for all i. So $x \in \bigcap_{i \in I} F_i$.

Corollary: (a) $\varnothing$ and X are open.
(b) If U_1 and U_2 are open, then so is $U_1 \cap U_2$.
(c) If U_i is open for all $i \in I$ then the union $\bigcup_{i \in I} U_i = \{x \in X \mid x \text{ is in some } U_i\}$ is open.

The **closure** $\bar{A}$ of a subset A is defined by:

$$\bar{A} = \{x \in X \mid \text{there exists a sequence in A with limit } x\}.$$

In order for $\bar{A}$ to always be closed, we require an extra property of our sequence space:
(5) if $t: N \times N \to X$ is a double sequence in X, if the sequence $t_{m0}, t_{m1}, t_{m2}, \ldots$ has limit s_m for each $m \in N$, and if the sequence $s_0, s_1, s_2, \ldots$ has limit x then there exist strictly increasing functions h and $k : N \to N$ such that the sequence $t_{h(0), k(0)}, t_{h(1),k(1)}, t_{h(2),k(2)}, \ldots$ has limit x.

Proposition: Premetric spaces satisfy (5).

Proof: For each m, there exists N_m such that $d(t_{mn}, s_m) < \frac{1}{m}$ for all $n \geq N_m$. We can choose the N_m to be strictly increasing in m. For each r, there is an M_r such that $d(s_m, x) < \frac{1}{r}$ for $m \geq M_r$. Then $d(tM_r,N_{Mr}, x) \leq d(tM_r,NM_r, s_{Mr}) + d(sM_r, x) < \frac{1}{M_r} + \frac{1}{r} \leq \frac{2}{r}$.

Let $h(r) = M_r$ and $k(r) = NM_r$. It follows that $(t_{h(n),k(n)})$ has limit x.

Proposition: If (5) holds then closures are closed.

Proof: Suppose s is a sequence in $\bar{A}$ with limit x. Then there is a sequence t_m in A with limit s_m. By (5) we have a sequence $t_{h(0),k)0)}, t_{h(1),k(1)}, t_{h(2),k(2)}, \ldots$ in A with limit x. So $x \in \bar{A}$. Hence $\bar{A}$ is closed.

In this case, $\bar{A}$ is the smallest closed subset of X containing A. That is, if $A \subseteq F$ with F closed then $\bar{A} \subseteq F$.

Let $f : X \to Y$ be a function between spaces X and Y.

Definition: f is **continuous at x** $\in X$ when, for all sequences s in X with limit x, the sequence $f(s)$ in Y has limit $f(x)$. We call f continuous when it is continuous at each $x \in X$.

Proposition: If $f : X \to Y$ is **continuous** at $x \in X$ and $g : Y \to Z$ is continuous at $f(x) \in Y$ then $g \circ f$ is continuous at x.

Proof: If s is a sequence in X with limit x then by continuity of f at x, we know that $f(x)$ is a limit for $f(s)$. By continuity of g at $f(x)$, it follows that $g(f(x)) = (g \circ f)(x)$ is a limit for $g(f(s)) = (g \circ f)(s)$. So $g \circ f$ is continuous at x.

Corollary: Composites of continuous functions are continuous.

For any function $f : X \to Y$, we write
$fA = \{f(x) \mid x \in A\} \subseteq Y$ for $A \subseteq X$, and
$f^{-1}B = \{x \in X \mid f(x) \in B\} \subseteq X$ for $B \subseteq Y$.

Theorem: For a function $f : X \to Y$ between spaces, the following conditions are equivalent:
(a) f is continuous;
(b) for all $x \in X$ and all nhds N of $f(x)$, there exists a nhd M of x with $fM \subseteq N$;
(c) if V is open in Y then $f^{-1}V$ is open in X;
(d) if G is closed in Y then $f^{-1}G$ is closed in X;

(e) for all $A \subseteq X$, $f\bar{A} \subseteq \overline{fA}$.

Proof: (a) $\Rightarrow$ (b): Take N as in (b) and put $M = f^{-1}N$ so that we have $fM \subseteq N$. It remains to show that M is a nhd of x. Let s be a sequence with limit x. Using (a) we know that $f(s)$ has limit $f(x)$. So $f(s)$ is eventually in N, Hence $f(s_n) \in N$ for n big enough. Thus $s_n \in M$ for n big enough. So s is eventually in M.

(b) $\Rightarrow$ (c): Take V open in Y. According to a previous proposition we need to see that $f^{-1}V$ is a nhd of each $x \in f^{-1}V$. But V is a nhd of $f(x)$ since $f(x) \in V$ and V is open. By (b), there is a nhd M of x with $fM \subseteq V$. So $x \in M \subseteq f^{-1}V$. So $f^{-1}V$ is a nhd of x.

(c) ⇒ (d): Suppose G is a closed subset of Y. Then $Y - G$ is open. By (c) $f^{-1}(Y - G) = X - f^{-1}G$ is open. Hence $f^{-1}G$ is closed.

(d) ⇒ (e): Take any $A \subseteq X$. Then $f\overline{A}$ is closed in Y. By (c), $f^{-1}\overline{fA}$ is closed. But $fA \subseteq \overline{fA}$, so $A \subseteq f^{-1}\overline{fA}$. But $\overline{A}$ is the smallest closed subset containing A. Hence $\overline{A} \subseteq f^{-1}\overline{fA}$ and so $f\overline{A} \subseteq \overline{fA}$.

(e) ⇒ (a): Suppose s is a sequence in X with limit x. We want to prove that $f(s)$ has limit $f(x)$. Suppose not. Then, by an axiom, there exists a subsequence t of s such that no subsequence of $f(t)$ has limit $f(x)$. Let $A = \{t_0, t_1, t_2, \ldots\}$. Since x is a limit of s, it is also a limit of t, so $x \in \overline{A}$. By (e), $f(x) \in \overline{fA}$. So there exists a sequence in $fA = \{f(t_0), f(t_1), f(t_2), \ldots\}$with limit $f(x)$. Any sequence in fA must have a subsequence of the form $f(r)$ where r is a subsequence of t. So we have a subsequence r of t for which $f(r)$ has limit $f(x)$. This contradicts the property of t.

Corollary: For any space X (satisfying (i) – (iv)) a sequence s in X has limit x in X if and only if s is eventually in every open subset containing x.
Proof: We already know "only if" from the definition of an open subset. Conversely suppose that s is eventually in every open U containing x.

We define a function $f : \hat{\mathbf{N}} \to X$ by
$f(n) = s_n$ for $n \in N$ and
$f(\infty) = x$.

It is easy to see that a subset S of $\hat{\mathbf{N}}$ is open if and only if either S does not contain ∞, or $\hat{\mathbf{N}} - S$ is finite. For any $U \subseteq X$, we see that, if $x \notin U$ then $f^{-1}U = \{n \in \mathbf{N} \mid s_n \in U\}$ which is always open. However if $x \in U$, $f^{-1}U = \{n \in \mathbf{N} \mid s_n \in U\} \cup \{\infty\}$ which is open as s is eventually in U. We want to apply condition (c) of the theorem which becomes the requirement that each open U containing x should have $f^{-1}U$ open. But this is exactly our assumption that s should eventually be in every such U. So f is continuous. Therefore f preserves limits of sequences. We have the sequence 0, 1, 2, 3, … in $\hat{\mathbf{N}}$ with limit ∞. Applying f we see that $s_0, s_1, s_2, s_3, \ldots$ has limit x in X.

21.5 Subspaces

Let A be a subset of a space X. We can make A into a space by defining a sequence s in A to have limit $a \in A$ when s has limit a in X. In this case A is called a **subspace** of X. Clearly the inclusion function $i : A \to X$, defined by $i(a) = a$ for all $a \in A$, is continuous. In fact we have provided sequences in A with the fewest possible limits so as to ensure that i is continuous.

Exercises
(1) Show that a subset M of A is a nhd of $a \in A$ if and only if there exists a nhd N of a in X such that $M = N \cap A$.
(2) Show that a subset U of A is open in A if and only if there exists an open subset V of X such that $U = V \cap A$.
(3) Explain why the interval [0, ½) is an open subset of the subspace [0, 1] of $\mathbf{R}$.
(4) Show that $\mathbf{N}$ is discrete as a subspace of $\mathbf{R}$.
(5) Let A be a subset of a metric space X. Regard A as a metric space by restricting the metric of X to A. Show that the space A defined by this metric is the same as A as a subspace of X.

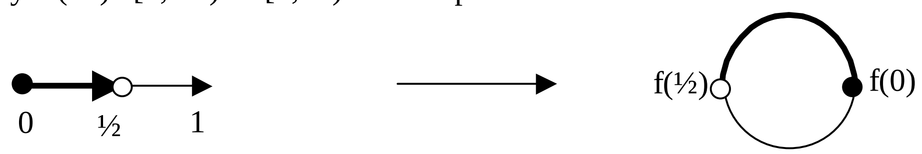
Examples: $\mathbf{I} = [0, 1] = \{x \in R \mid 0 \le x \le 1\}$ is a subspace of $\mathbf{R}$.
$\mathbf{I}^2 = \{(x, y) \in \mathbf{R}^2 \mid 0 \le x, y \le 1\}$ and
$\mathbf{S}^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ are subspaces of $\mathbf{R}^2$.
$\mathbf{S}^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is a subspace of $\mathbf{R}^3$.

21.6 More on Continuity

Proposition: The functions "add" and "multiply" $\mathbf{R}^n \to \mathbf{R}$, the diagonal function $\Delta : \mathbf{R} \to \mathbf{R}^n$ taking x to $(x, x, \dots, x)$ and, for each $a \in \mathbf{R}$ the function $(a, -) : \mathbf{R} \to \mathbf{R}^2$ taking x to (a, x), are all continuous.
If $f_1, \dots f_n : \mathbf{R} \to \mathbf{R}$ are continuous so is $f_1 \times \dots \times f_n : \mathbf{R}^n \to \mathbf{R}^n$ taking $(x_1, \dots, x_n)$ to $(f_1(x_1), \dots, f_n(x_n))$.
Proof: The first two functions are continuous since a sum and product of convergent sequences in $\mathbf{R}$ converge to the sum and product of the limits. The other functions are continuous since a sequence in $\mathbf{R}^n$ converges if and only if the component sequences do in R.
Corollary: Polynomials are continuous functions from R to R. That is, for all $a_0, a_1, \dots, a_n \in R$ the function $a : \mathbf{R} \to \mathbf{R}$ defined by $a(x) = a_0 + a_1x + \dots + a_nx^n$ is continuous.

Definition: A function $f : X \to Y$ between spaces is called a homeomorphism when it is continuous with a continuous inverse. Write $X \cong Y$ when there is such a homeomorphism.
Note: A continuous function may have an inverse which is not continuous.

Examples
(1) Let $\mathbf{R}$ be the usual space of real numbers and let $\mathbf{R}_d$ be the real numbers as a discrete space. The identity function $\mathbf{R}_d \to \mathbf{R}$ taking x to x is continuous since eventually constant sequences certainly converge in $\mathbf{R}$. It is not a homeomorphism since there are convergent sequences in $\mathbf{R}$ that converge and yet are not eventually constant.
(2) Let $f : [0, 1) \to \mathbf{S}^1$ be the function $f(t) = (\cos 2\pi t, \sin 2\pi t)$.
Then f is continuous and has an inverse. However f is not a homeomorphism since $[0, ½)$ is open in $[0, 1)$ yet $(f^{-1})^{-1}[0, ½) = f[0, ½)$ is not open in $\mathbf{S}^1$.



(3) Any two open intervals in R are homeomorphic. For open intervals of finite length we have the homeomorphism $f : (0, 1) \to (a, b)$ defined by $f(t) = a(1 - t) + bt$. and where $f^{-1}(s) = \dfrac{s - a}{b - a}$.
For the open interval $\mathbf{R} = (-\infty, \infty)$ we have the homeomorphism $g : \mathbf{R} \to (-1, 1)$ defined by $g(r) = \dfrac{r}{1 + |r|}$ where $g^{-1}(s) = \dfrac{s}{1 - |s|}$

We can show spaces are homeomorphic by exhibiting a homeomorphism, but how do we show spaces are *not* homeomorphic? Why is there no homeomorphism between the cylinder and the Möbius Band? We will develop techniques for this by examining properties of spaces that are invariant under homemorphism. We already know of two such properties: Hausdorffness and compactness.

Example: We will prove later that $\mathbf{I} = [0, 1]$ is compact. Clearly $\mathbf{R}$ is not compact since the sequence 0, 1, 2, 3, … has no convergent subsequence. So $\mathbf{R}$ is not homeomorphic to $\mathbf{I}$.