11. THE ALEXANDER MODULE

11.1 Modules over the Ring of Rational Laurent Polynomials

There's an even better invariant for a knot than the Alexander group – the Alexander module. But before we can discuss it we need to learn some more algebra.

We all know about polynomials. A polynomial is a formal expression of the form:

$$a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0.$$

where the i's are the coefficients and "t" is an "indeterminate. (We'll use "t" rather than the traditional "x".)

The coefficients an come from any field, but we'll restrict ourselves to the case where the coefficients are rational numbers. These rational polynomials form a ring under the usual operations of addition and multiplication of polynomials. A *ring* is a mathematical structure with two binary operations + and × satisfying a whole bunch of axioms. In fact the ring **Q**[t] of rational polynomials comes very close to being a field. You should know the field axioms from having done a linear algebra course. The one field axiom that breaks down for rational polynomials is the axiom that says that every non-zero element has an inverse under multiplication. The only rational polynomials that do have multiplicative inverses in **Q**[t] are the non-zero constant polynomials.

Now there's no good reason why we can't include negative powers of t. Of course expressions such as $2t + 3 + \frac{1}{t^2}$ aren't polynomials, but we can still add and multiply them as we do ordinary polynomials. We call them Laurent polynomials.

A **Laurent polynomial** is a formal expression of the form:

$$a_n t^n + a_{n-1} t^{n-1} + \ldots + a_m t^m$$

where m, n are integers, possibly negative, with $n \geq m$. As with ordinary polynomials the a_i are the coefficients and the "t" is an indeterminate. We'll be considering rational Laurent polynomials, where the coefficients are rational numbers and, of course, they include the ordinary rational polynomials.

Example 1: $2t + 3 + \frac{1}{t^2}$ and $t^2 + 4t - \frac{1}{2t}$ are Laurent polynomials.

Their sum is $(2t + 3 + \frac{1}{t^2}) + (t^2 + 4t - \frac{1}{2t}) = t^2 + 6t + 3 - \frac{1}{2t} + \frac{1}{t^2}$ and their product is

$(2t + 3 + \frac{1}{t^2})\,(t^2 + 4t - \frac{1}{2t}) = 2t^3 + 11t^2 + 12t + \frac{5}{2t} - \frac{1}{2t^3}$.

The set of all rational Laurent polynomials in t is denoted by **Q**(t). It contains the set of all (ordinary) rational polynomials **Q**[t]. Like **Q**[t] it comes close to being a field, but again it's the lack of inverses under multiplication that prevent it from being a field.

The only rational Laurent polynomials with multiplicative inverses are those with exactly one non-zero coefficient, of the form at^n where $a \neq 0$.

Now an abelian group is rather similar to a vector space. Remember that in a vector space you can add any two vectors and the axioms for addition are precisely those for an abelian group. But in a vector space there's the operation of multiplication by a scalar, with several axioms regulating this operation. These additional axioms, for a vector space V over the field F, are:

- $\lambda\mathbf{v} \in V$ for all $\mathbf{v} \in V$ and $\lambda \in F$;
- $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ for all $\mathbf{v} \in V$ and $\lambda, \mu \in F$;
- $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in F$;
- $1v = v$ for all $v \in V$.

But abelian groups don't have scalars – at least there was no mention of them in the definition. This is true, but remember that we defined ng for any integer n and any group element g. We can therefore consider the integers as being scalars. But **Z** isn't a field. That's also true. That's why abelian groups aren't vector spaces. But notice that all four of the above vector space axioms hold in an abelian group. Let's write them out again, this time using notation that's more appropriate for an abelian group G.

- $ng \in G$ for all $g \in G$ and $n \in \mathbf{Z}$;
- $(m + n)g = mg + ng$ for all $g \in G$ and $m, n \in \mathbf{Z}$;
- $n(g + h) = ng + nh$ for all $g, h \in G$ and $n \in \mathbf{Z}$;
- $1g = g$ for all $g \in G$.

What we're going to do is to extend our ring of scalars to include not just integers, but all rational Laurent polynomials. And, as with our abelian groups, we're going to consider formal linear combinations. But the coefficients will come from **Q**(t).

So a typical formal linear combination will have the form:

$$a_1(t)x_1 + \ldots + a_n(t)x_n$$

where the coefficients, $a_i(t)$ are rational Laurent polynomials and where the x_i are indeterminates.

We'll denote the set of all such formal linear combinations of the indeterminates $x_1, \ldots, x_n$, over **Q**(t), by $\langle x_1, \ldots x_n\rangle$. This is a similar notation to $[x_1, \ldots, x_n]$ for the abelian group case, where the coefficients are just integers.

We can add such expressions together in the obvious way, and even multiply them by a scalar from **Q**(t). Moreover all the vector space axioms hold, apart from the small detail that the ring of scalars, **Q**(t), isn't a field. We call such structures **Q(t)-modules**.

Example 2: If $a = (2t + 3 + \frac{1}{t^2})\,x + (t^2 + 4t - \frac{1}{2t})\,y$ and $b = (t^3 + t + 1)\,x + \frac{1}{2t}\,y$ then $a, b \in \langle x, y\rangle$.

$a + b = (t^3 + 3t + 4 + \frac{1}{t^2})\,x + (t^2 + 4t)\,y$ and

$$(t - \frac{1}{t})\,a = (t - \frac{1}{t})\,(2t + 3 + \frac{1}{t^2})\,x + (t - \frac{1}{t})\,(t^2 + 4t - \frac{1}{2t})\,y$$

$$= (2t^2 + 3t - 2 - \frac{2}{t} - \frac{1}{t^3})\,x + (t^3 + 4t^2 - t - \frac{9}{2} + \frac{1}{2t^2})\,y.$$

We can't define ab unless we have a definition of xy etc which we don't.

Just as we have abelian groups defined by means of generators and relations so we have **Q**(t)-modules defined in this way.

Example 3: $\langle x, y \mid tx + \frac{1}{t}y = 0\rangle$ is a **Q**(t)-module. But since we can express the relation $tx + \frac{1}{t}y = 0$ in the form $y = -t^2x$, the generator y is redundant. We can remove y and the relation that defines it. So we are left with $\langle x\rangle$ with no relations. We call this a **cyclic Q**(t)-module.

Example 4: $\langle x \mid (t^2 + 2t + 3)x = 0\rangle$ is also a cyclic **Q**(t)-module (one generator). But this time we have one relation. The elements have the form $a(t)x$ where $a(t) \in \mathbf{Q}(t)$. But, from the relation $(t^2 + 2t + 3)x = 0$ we can write $t^nx = -2t^{n-1}x - 3t^{n-2}x$ for all $n \geq 2$. This means that we can express terms involving $t^2x, t^3x, \ldots$ in terms of lower powers of t and ultimately all terms involving t^nx where $n \geq 2$ can be written in terms of just tx and x.

Similarly we can rewrite $(t^2 + 2t + 3)\,x = 0$ in the form $\frac{1}{t^n}x = -\frac{1}{3t^{n-2}}x - \frac{2}{3t^{n-1}}x$ for all $n \geq 1$. This means that we can express terms involving $\frac{1}{t}x, \frac{1}{t^2}x, \ldots$ in terms of higher powers of t and ultimately all terms involving $\frac{1}{t^n}x$ where $n \geq 1$ can be written in terms of just tx and x.

This all means that $\langle x \mid (t^2 + 2t + 3)x = 0\rangle = \{(a + bt)x \mid a, b \in \mathbf{Q}\}$. As a Q(t) module it has one generator and so is cyclic. As a vector space over **Q** it has dimension 2, with a basis {x, tx}.

Theorem 1: If a(t) is a rational (ordinary) polynomial of degree n then $M = \langle x \mid a(t)x = 0\rangle$ is a cyclic **Q**(t)-module. Every element can be expressed in the form $(c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \ldots c_1x + c_0)x$ and so, as a vector space over **Q**, has dimension n.

Proof: Suppose $a(t) = a_nt^n + a_{n-1}t^{n-1} + \ldots a_1t + a_0$ where $a_n \neq 0$.
We can use the equation $(a_nt^n + a_{n-1}t^{n-1} + \ldots a_1t + a_0)x = 0$ express t^nx in terms of $x, tx, \ldots, t^{n-1}x$. Multiplying this equation by t we get $(a_nt^{n+1} + a_{n-1}t^n + \ldots + a_1t^2 + a_0t)x = 0$. This can be used to express $t^{n+1}x$ in terms of $tx, t^2x, \ldots, t^{n-1}x, t^nx$. But since t^nx can be expressed in terms of lower powers of t we can ultimately express t^rx, for any positive r, in terms of $x, tx, t^2x, \ldots, t^{n-1}x$.

Now we may assume that $a_0 \neq 0$, for if not we merely multiply the above relation by a suitable power of t^{-1} until it is.

Dividing this relation by t we can express $\frac{1}{t}x$ in terms of $x, tx, \ldots, t^{n-1}x$. Dividing by t^2 we can express $\frac{1}{t^2}x$ in terms of $\frac{1}{t}x, x, tx, \ldots, t^{n-2}x$. But since $\frac{1}{t}x$ can be expressed in terms of $x, tx, \ldots, t^{n-1}x$ we can express $\frac{1}{t^2}x$ similarly. Ultimately we can express $\frac{1}{t^r}x$, for any positive r, in terms of $x, tx, t^2x, \ldots, t^{n-1}x$..

This shows that M is spanned, as a vector space over **Q**, by $x, tx, t^2x, \ldots t^{n-1}x$.

By analogy with integers we call the Q(t)-module the ring of rational Laurent polynomials modulo a(x) and denote it by the symbol $\mathbf{Q}(t)_{a(t)}$.

Example 5: $\langle x \mid (t^3 - t^2 - t + 2)x = 0\rangle$.
From $(t^3 - t^2 - t + 2)x = 0$ we conclude that

$t^3x = t^2x + tx + 2x$

$\therefore t^4x = t^3x + t^2x + 2tx = (t^2x + tx + 2x) + t^2x + 2tx = 2t^2x + 3tx + 2x$

$\therefore t^5x = 2t^3x + 3t^2x + 2tx = 2(t^2x + tx + 2x) + 3t^2x + 2tx = 5t^2x + 4tx + 4x$

…………………………

From $(t^3 - t^2 - t + 2)x = 0$ we conclude that $2x = -t^3x + t^2x + tx$

so $x = -½t^3x + ½t^2x + ½tx$.

$\therefore \frac{1}{t}x = -½t^2x + ½tx + ½x$.

$\therefore \frac{1}{t^2}x = -½tx + ½x + ½.\frac{1}{t}x = -½tx + ½x + ½(-½t^2x + ½tx + ½x) = -¼t^2x - ¼tx + ¾x$

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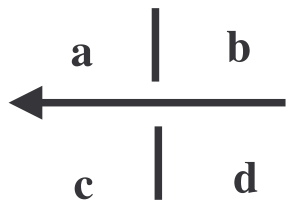
11.2 The Alexander Module of an Oriented Link

Suppose L is a knot. Take a projection of the link and assign a generator to each face. (So far we are proceeding as for the face group.) These are the generators for our module.

Now choose an orientation for each component of the link, marking these by arrows. We obtain a relation at each of the crossings as follows. View the crossing so that the orientation of the overpass is from right to left. Then, if a, b, c and d are the generators corresponding to the faces surrounding the crossing, as indicated in the following diagram, we take the relation

$$\mathbf{t\,(a + b) + (c + d) = 0.}$$



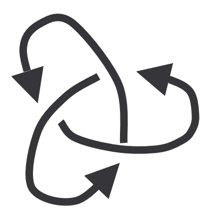
If $t = -1$ this becomes the corresponding relation for the face group and the orientation doesn't matter. However, for the Alexander module, we treat t as an indeterminate. The resulting algebraic structure is a module over $\mathbf{Q}(t)$, the ring of rational Laurent polynomials and we call it the **face module** of the oriented link.

As we did with face groups we put the generators for two adjacent faces equal to zero and introduce generators only as required. So long as we have labels for 3 out of 4 faces around a crossing we can express the generator in terms of the 4th face in terms of them. Otherwise we introduce a new generator.

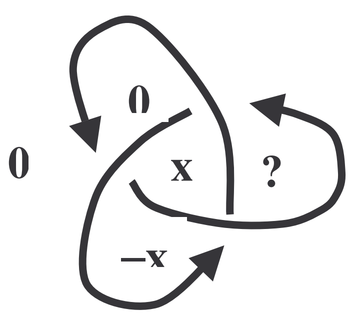
For fairly oriented link we would only have to use one generator, say x, and there'd be just one relation of the form $p(t)x = 0$, where $p(t)$ is a Laurent polynomial. But since t^{-1} is a Laurent polynomial we're entitled to multiply or divide $p(t)$ by powers of t. So we can arrange for $p(t)$ to be an ordinary polynomial with non-zero constant term. The module in this case would be the ring of rational Laurent polynomials modulo $p(t)$, which we denote by $\mathbf{Q}(t)_{p(t)}$.

Theorem 2: The Alexander polynomial is an invariant of an oriented link.
Proof: To show this we simply follow the procedure detailed in Theorem 1 of chapter 10: we show that the Alexander module is unchanged by each of the three Reidemeister moves.

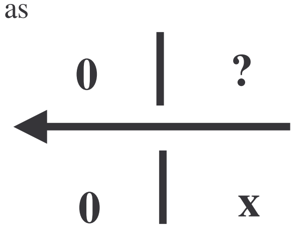
Example 6: Find the Alexander module of the following oriented trefoil knot.



Solution:
Using the crossing adjacent to the two zeros we assign the label $-x$.

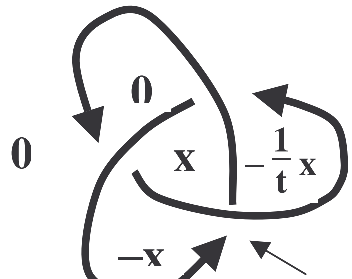


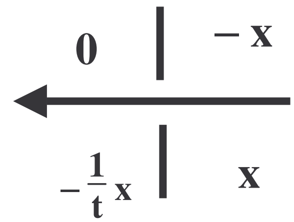
We view the topmost crossing as



which gives $t(0 + ?) + 0 + x = 0$ and hence $? = -\frac{1}{t}x$.

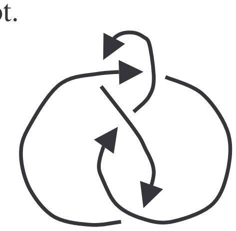
At the remaining crossing we get the relation $t(0 - x) - \frac{1}{t}x + x = 0$. This can be simplified to $\left(t - 1 + \frac{1}{t}\right)x$.

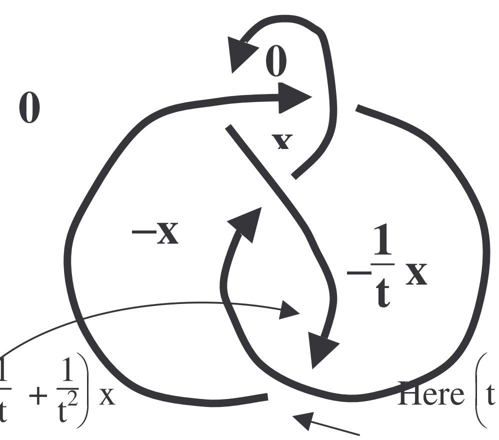




Since t has an inverse in the ring of Laurent polynomials we are permitted to multiply or divide by powers of t. Here we choose to multiply by t, obtaining $(t^2 - t + 1)x = 0$.
So the Alexander module of this oriented trefoil knot is $\mathbf{Q}(t)_{t^2 - t + 1}$.

Example 7: Find the Alexander module of the following oriented figure 8 knot.
Solution:

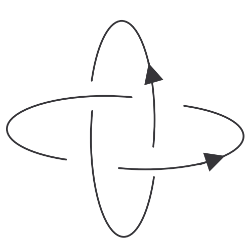


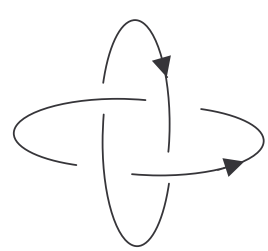


$\left(1 - \frac{1}{t} + \frac{1}{t^2}\right)x$ Here $\left(t - 3 + \frac{1}{t}\right)x = 0$.
$\therefore (t^2 - 3t + 1)x = 0$.
The Alexander module of this oriented figure 8 knot is thus $\mathbf{Q}(t)_{t^2 - 3t + 1}$.

The following example shows that while the Alexander module is an invariant of an *oriented* link it is not an invariant of the link itself.

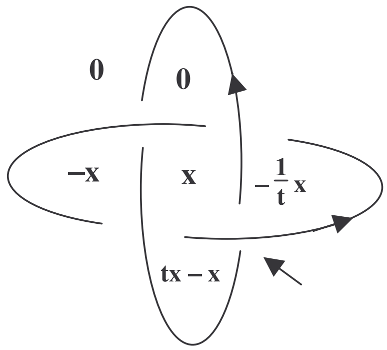
Example 8: Find the Alexander modules of the following oriented links:
(a) (b)





Solution:

(a)



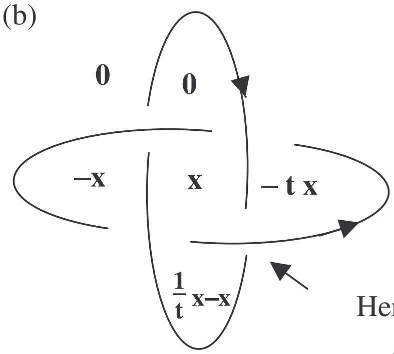
Here $t(0 + tx - x) - \frac{1}{t}x + x = 0$

$\therefore$ $\left(t^2 - t - \frac{1}{t} + 1\right)x = 0$

$\therefore$ $(t^3 - t^2 + t - 1)\,x = 0$.

The Alexander module of this oriented link is thus $\mathbf{Q}_{t^3 - t^2 + t - 1}$.

(b)

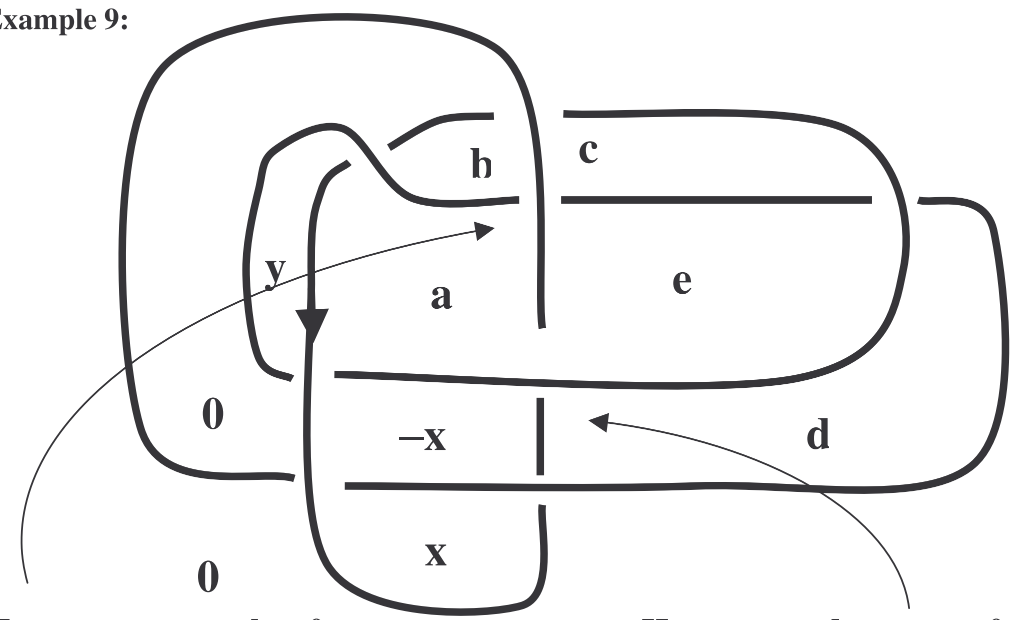


Here $t\left(\frac{1}{t}x - x\right) + x - tx = 0$

$\therefore$ $(2t - 2)x = 0$.

The Alexander module of this oriented link is thus $\mathbf{Q}_{2t - 2}$.

Example 9:



Here tc + te + a + b = 0. **Here −tx + td + a + e = 0.**

$ty + a - x = 0$
$\therefore\ a = x - ty$

$tb + a + y = 0$
$\therefore\ b = y - \frac{1}{t}x - \frac{1}{t}y$

$tc + b = 0$
$\therefore\ c = -\frac{1}{t}y + \frac{1}{t^2}x + \frac{1}{t^2}y$

$tx - x + d = 0$
$\therefore\ d = x - tx$

$td + c + e = 0$
$\therefore e = -tx + t2x + \frac{1}{t}y - \frac{1}{t^2}y - \frac{1}{t^2}x$

Since $tc + te + a + b = 0$ we have $(-t^2 + t^3 + 1 - \frac{1}{t})x + (1 - t - \frac{1}{t})y = 0$ and so, multiplying by t, $(t^4 - t^3 + t - 1)x - (t^2 - t + 1)y = 0$.

Since $-tx + td + a + e = 0$ we can show that $(t^3 - t^2 + 1)x + (t^3 - t + 1)y = 0$.

The Alexander module is therefore

$$\begin{bmatrix} t^3 - t^2 + 1 & t^3 - t + 1 \\ t^4 - t^3 + t - 1 & -t^2 + t - 1 \end{bmatrix}$$
$$\cong \begin{bmatrix} t^3 - t^2 + 1 & t^3 - t + 1 \\ t^4 - t^2 + t & t^3 - t^2 \end{bmatrix} \ (R_2 + R_1)$$
$$\cong \begin{bmatrix} t^3 - t^2 + 1 & t^3 - t + 1 \\ t^3 - t + 1 & t^2 - t \end{bmatrix} \ (R_2 \div t)$$
$$\cong \begin{bmatrix} t^3 - t^2 + 1 & t^2 - t \\ t^3 - t + 1 & -t^3 + t^2 - 1 \end{bmatrix} \ (C_2 - C_1$$
$$\cong \begin{bmatrix} 1 & t^2 - t \\ t^4 + 1 & -t^3 + t^2 - 1 \end{bmatrix} \ (C_1 - tC_2)$$

$$\cong \begin{bmatrix} 1 & 0 \\ t^4 + 1 & -t^6 + t^5 - t^3 + t - 1 \end{bmatrix} \ (C_2 - (t^2 - t)C$$
$$\cong \begin{bmatrix} 1 & 0 \\ t^4 + 1 & t^6 - t^5 + t^3 - t + 1 \end{bmatrix} \ (C_2 \to -C_2)$$
$$\cong \begin{bmatrix} 1 & 0 \\ 0 & t^6 - t^5 + t^3 - t + 1 \end{bmatrix} \ (R_2 - (t^4 + 1)R_1)$$
$$\cong [t^6 - t^5 + t^3 - t + 1]$$
$$\cong \mathbf{Q}(t)_{t^6 - t^5 + t^3 - t + 1}$$

11.3 The Alexander Polynomial of an Oriented Link

If p(t) and q(t) are Laurent polynomials then the modules $\mathbf{Q}_{p(t)}$ and $\mathbf{Q}(t)_{q(t)}$ are isomorphic if and only if $p(t) = \pm q(t)t^n$ for some integer n. If we normalise our polynomials so that they become ordinary polynomials with positive leading coefficients, then $\mathbf{Q}_{p(t)}$ and $\mathbf{Q}(t)_{q(t)}$ are isomorphic if and only if p(t) = q(t). In such cases the polynomial is as good an invariant as the module. We call this polynomial the **Alexander polynomial** of the oriented link.

In more complicated cases we would be forced to introduce more than one generator. In such cases it may be very difficult to decide whether or not to modules are isomorphic. However we can still define the Alexander polynomial.

If M is the matrix of coefficients of the relations, for a module given by n generators and n relations, we can calculate the determinant of M. Now remember that the coefficients in the relations will be Laurent polynomials (although we can arrange for them to be ordinary polynomials by multiplying by suitable powers of t). So the determinant of M will also be a Laurent polynomial. If we multiply by a suitable power of t so that it becomes an ordinary polynomial with non-zero constant term, and by −1 if necessary to make the leading coefficient positive, then this polynomial is called the determinant of the module. It can be shown that it is an invariant of the module, that is, if two presentations give isomorphic modules, the determinants are equal. (But the converse does not hold).

The **Alexander polynomial** of an oriented link is the determinant of its Alexander module. It is an invariant of the oriented link. If two oriented links are equivalent their Alexander modules are isomorphic and so their Alexander polynomials are equal. But, as usual, the converse is not true.

The Alexander polynomial plays the same role with Alexander modules as the Alexander number does to Alexander group. It's a single object (polynomial or integer) that's a convenient invariant that can be used in place of the algebraic structure (module or group). However the single object is not as powerful an invariant as the algebraic structure.

Putting t = −1 the Alexander module collapses down to the Alexander group and, if the Alexander polynomial is p(t) the Alexander number is p(−1).

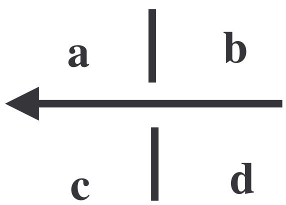
Example 10: The Alexander polynomials of the oriented links in example 8 are $t^3 - t^2 + t - 1$ and $2t - 2$ respectively. Since both are the same link, with different orientations, this shows that the Alexander polynomial is not an invariant of a link. Notice that substituting t = −1 in each of these polynomials, and then taking the absolute value, we get 4 in each case. This reflects the fact that 4 is the Alexander number for this link.

If a link has n connected components there are 2^n possible orientations, each with its own Alexander polynomial (though some of these may be the same). So to use the Alexander polynomial as an invariant for a link we would have to obtain the set of Alexander polynomials for all possible orientations. Then, if two links had disjoint sets of Alexander polynomials we could conclude that the links were not equivalent. This is very unwieldy, though the following theorem helps a little.

Theorem 3: If the Alexander polynomial of an oriented link is $a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$ then the Alexander polynomial of the oriented link obtained by reversing the orientation of every component is $\pm(a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n)$.

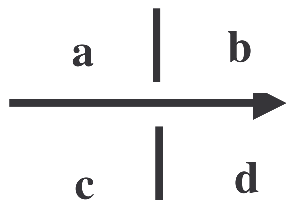
Proof: Reversing the orientation of every component is equivalent to replacing t by $\frac{1}{t}$

$t(a + b) + (c + d) = 0$

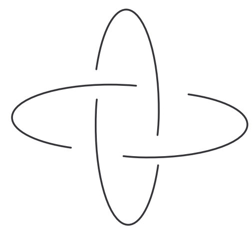


$t(c + d) + (a + b) = 0$

i.e. $\frac{1}{t}(a + b) + (c + d) = 0$



Example 11: The set of Alexander polynomials for the link is $\{t^3 - t^2 + t - 1, 2t - 2\}$. We computed two of the four possible orientations in example 10 and the other two are the same by Theorem 3.



Because of the problem with orientation the Alexander polynomial (and the Alexander module) are not very satisfactory tools for use with links in general. For knots, however, the situation is very much better.

We define a polynomial $a_n t^n + a_{n-1} t^{n-1} + \ldots.. + a_1 t + a_0$ to be **symmetric** if the coefficients are symmetric, that is, if $a_r = a_{n-r}$ for all r with $0 \le r \le n$.

Example 12: The polynomial $3t^8 - 17t^7 + 33t^5 + 5t^4 + 33t^3 - 17t + 3$ is symmetric.

The following theorem is an obvious consequence of Theorem 3.

Theorem 4: If the Alexander polynomial of one orientation of a knot is symmetric then it is also the Alexander polynomial of the other orientation and hence is an invariant of the knot itself.

The next theorem is far from obvious, and we omit the proof.

Theorem 5: The Alexander polynomial of every oriented knot is symmetric.
Corollary: The Alexander polynomial is an invariant for knots.

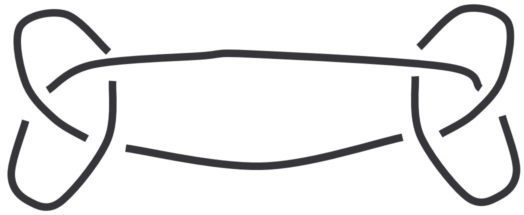
Example 11: The Alexander polynomial of the oriented link in example 9 is $t^6 + t^5 - t^3 + t + 1$**.**
We could have determined this prior to carrying out the row and column operations as follows:

$$\begin{vmatrix} t^3 + t^2 - 1 & t^3 - t - 1 \\ t^4 + t^3 - t - 1 & -t^2 - t - 1 \end{vmatrix} = (t^3 + t^2 - 1)(-t^2 - t - 1) - (t^3 - t - 1)(t^4 + t^3 - t - 1)$$

$$= (-t^5 - 2t^4 - 2t^3 + t + 1) - (t^7 + t^6 - t^5 - 3t^4 - 2t^3 + t^2 + 2t + 1) = -t^7 - t^6 + t^4 - t^2 - t.$$

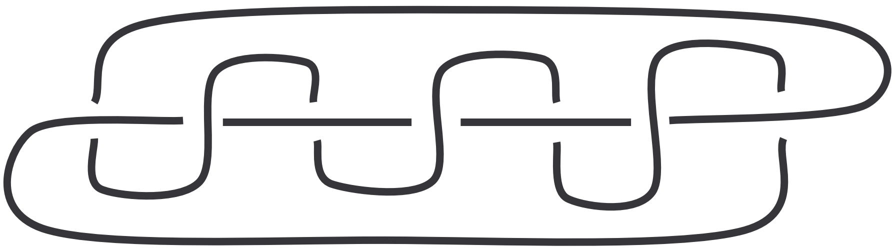
Dividing by $-t$, to normalize, we get $t^6 + t^5 - t^3 + t + 1$.

Example 12:



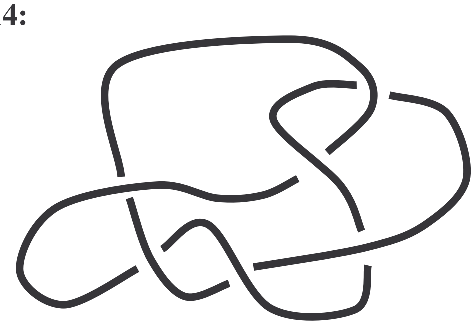
Alexander Module: $\begin{bmatrix} t^2 - t + 1 & 0 \\ 0 & t^2 - t + 1 \end{bmatrix}$.
Alexander Polynomial: $(t^2 - t + 1)^2$.
Alexander Group: $\mathbf{Z}_3 \oplus \mathbf{Z}_3$.
Alexander Number: 9.

Example 13:



Alexander Module: $\mathbf{Z}(t)_{t^8-t^7+t^6-t^5+t^4-t^3+t^2-t+1}$.
Alexander Polynomial: $t^8-t^7+t^6-t^5+t^4-t^3+t^2-t+1$.
Alexander Group: $\mathbf{Z}_9$.
Alexander Number: 9.

Example 14:



Alexander Module: $\mathbf{Z}(t)_{2t^2-5t+2}$.
Alexander Polynomial: $2t^2-5t+2$.
Alexander Group: $\mathbf{Z}_9$.
Alexander Number: 9.

Theorem 6: If $\alpha(t)$ is the Alexander polynomial of a knot then $\alpha(1) = \pm 1$.
Proof: The absolute value of $\alpha(1)$ is the order of the abelian group obtained by putting $t = 1$ in the Alexander module. This is equivalent to using the relation $a + b + c + d = 0$ at each crossing. Because these relations are symmetric in the generators involved the resulting group will not change if under- and over- passes are interchanged at some of the crossings. But by suitably changing over- and under- passes at certain crossings the knot can be transformed into the unknot. So this group will be the trivial group. Hence $|\alpha(1)| = 1$.