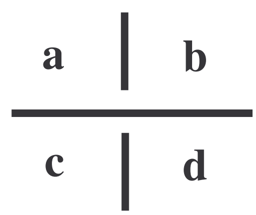
10. THE ALEXANDER GROUP OF A LINK

10.1 The Face Group

Let K be a knot and let M be a map for it. We define an abelian group for the knot in terms of generators and relations as follows:

Assign a generator to each face (including the outside). In fact we'll use the name of each face as the name of the corresponding generator.

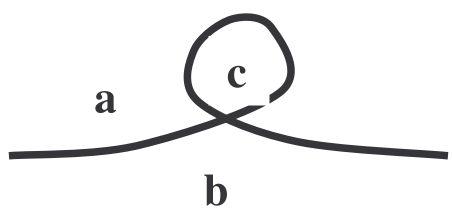
For each crossing



take the relation

$$a + b = c + d$$

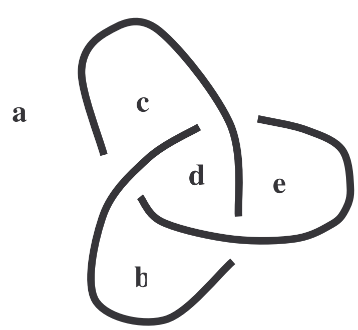
For a crossing of the form



this becomes a + c = a + b, whence b = c.

The abelian group with these generators and relations is called the face group of the map. Now if this group depended on the map, and not the knot itself, this would not be a very useful concept. However it does only depend on the knot and so we call it the **face group** F(K) of the knot.

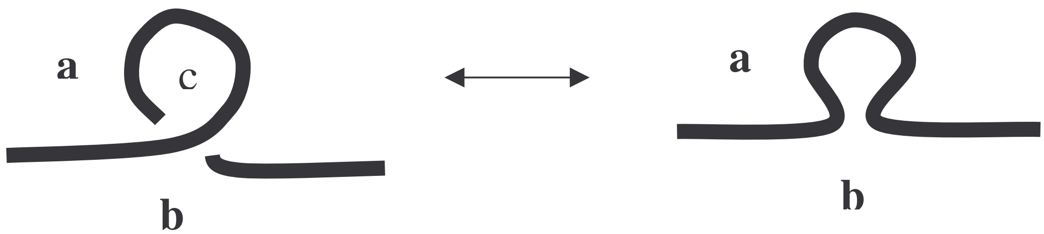
Example 1: F(trefoil) = = [a, b, c, d | a + c = b + d, c + d = a + e, d + e = a + b]



10.2 The Face Group is an Invariant of a Link

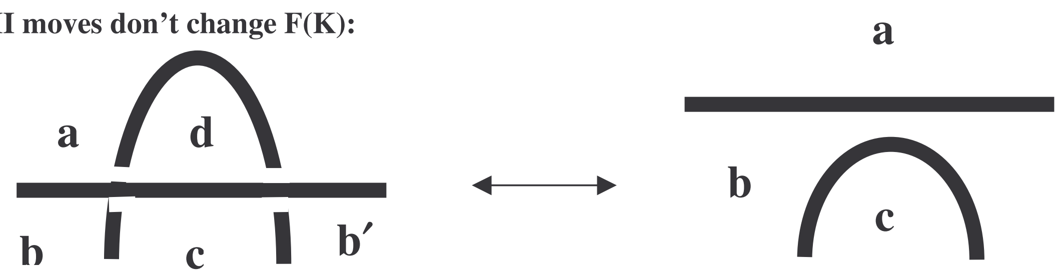
We'll now show that the three Reidemeister moves don't change the face group (up to isomorphism). Since any map of a link can be transformed to any other equivalent map by just using Reidemeister moves this shows that the face group depends only on the link. In other words, it is an invariant of the link.

Theorem 1: Each of the three Reidemeister moves leaves the face group unchanged, up to isomorphism.
Proof:
Type I moves don't change F(K):



The face groups will be the same except that the one on the left has an extra generator c and an extra relation $a + c = a + b$. A consequence of this relation is $c = b$. Since c isn't involved in any other relation we may remove c as a generator, as well as relation $c = b$. This group is then the face group of the knot on the right.

Type II moves don't change F(K):



The face groups of these two knots are identical except that the one on the left has two additional generators b' and d and two extra relations: $a + d = b + c$ and $a + d = b' + c$.

If the face group of the knot on the right is written as [… | …], the one on right would be […, b′, d | …, $a + d = b + c$, $a + d = b' + c$], where the generators and relations indicated by the dotted lines are identical for the two knots except that some of the occurrences of b in the relations of the first presentation may be replaced by b′ in the second. This is because the face called b in the right-hand knot is split into two faces, b and b′ in the map on the left.

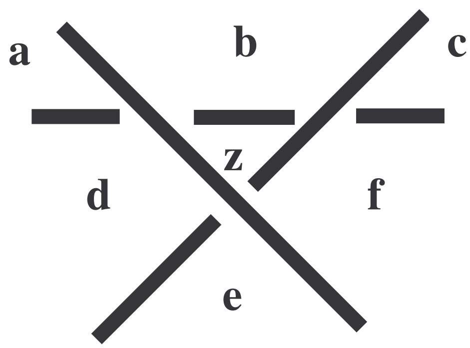
Now the relations:

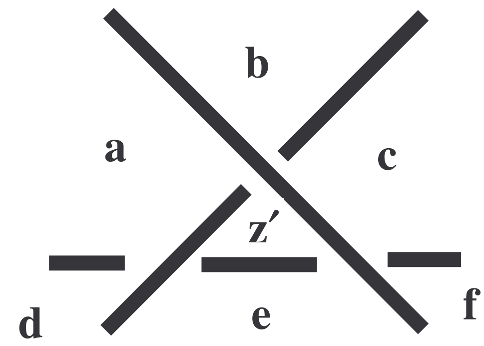
$$\left.\begin{array}{l} a + d = b + c \\ a + d = b' + c \end{array}\right\} \text{ are equivalent to } \left.\begin{array}{c} b' = b \\ d = b + c - a \end{array}\right\}.$$

We may omit the generator d since it is only involved in the relation $d = b + c - a$.

Hence […, b′, d | …, $a + d = b + c$, $a + d = b' + c$] ≅ […, b′ | …, $b = b'$]. We may now eliminate the generator b′ and the relation $b = b'$ provided that all occurrences of b′ in the other relations are replaced by b. But this gives us precisely the presentation for the right-hand knot.

Type III moves don't change F(K):





For the crossings not shown the relations are identical. The remaining three relations are:

LEFT HAND-MAP	RIGHT-HAND MAP
$\left.\begin{array}{l} a+d=b+z \\ b+z=c+f \\ z+f=e+d \end{array}\right\}$	$\left.\begin{array}{l} c+b=z'+a \\ c+f=z'+e \\ z'+e=a+d \end{array}\right\}$
i.e. $\left.\begin{array}{l} z=e-f+d \\ a+d=c+f \\ a+f=b+e \end{array}\right\}$	i.e. $\left.\begin{array}{l} z'=a+d-e \\ a+d=c+f \\ b+e=a+f \end{array}\right\}$

Since z and z′ don't appear in any other relation, and since we can express them in terms of the other generators, we may remove them from the set of generators, together with their defining relations which are the first relation in each set of three. Now both knots have identical generators and relations.

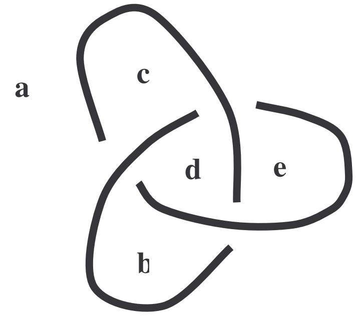
Since equivalent links can be obtained from one another by sequences of Reidemeister moves, it follows that the face group is an invariant of a link. This means that if L_1 and L_2 are two links and $F(L_1) \not\cong F(L_2)$ then $L_1 \not\approx L_2$. That is, non-isomorphic face groups means inequivalent links. But beware: *isomorphic face groups does not guarantee equivalent links*.

Example 2:
Suppose K_1, K_2 and K_3 are three knots. Suppose that:

$$F(K_1) \cong \mathbf{Z}_3 \oplus \mathbf{Z} \oplus \mathbf{Z};$$
$$F(K_2) \cong \mathbf{Z}_5 \oplus \mathbf{Z} \oplus \mathbf{Z};$$
$$F(K_3) \cong \mathbf{Z}_5 \oplus \mathbf{Z} \oplus \mathbf{Z};$$

Then we can conclude that $K_1 \not\cong K_2$ and $K_1 \not\cong K_3$. But we can't conclude that $K_2 \approx K_3$. Maybe they're equivalent — or maybe not.

Example 3: Find the face group of the trefoil knot:

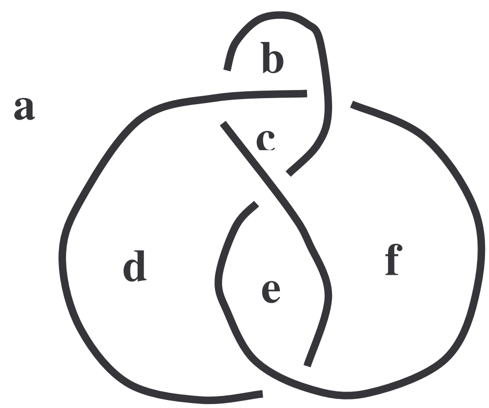


F(trefoil) ≈ [a, b, c, d, e | a + c = b + d, c + d = a + e, d + e = a + b]

$$\cong \begin{bmatrix} 1 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} -1 & 2 & 0 & -1 \\ -2 & 1 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & -2 & 0 & 1 \\ -2 & 1 & 0 & 1 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & -3 & 0 & 3 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 3 \end{bmatrix} \cong [-3, 0, 3] \cong [3, 0, -3] \cong [3, 0, 0] \cong \mathbf{Z}_3 \oplus \mathbf{Z} \oplus \mathbf{Z}.$$

Example 4: The face group of the figure 8 knot is isomorphic to $\mathbf{Z}_5 \oplus \mathbf{Z} \oplus \mathbf{Z}$.



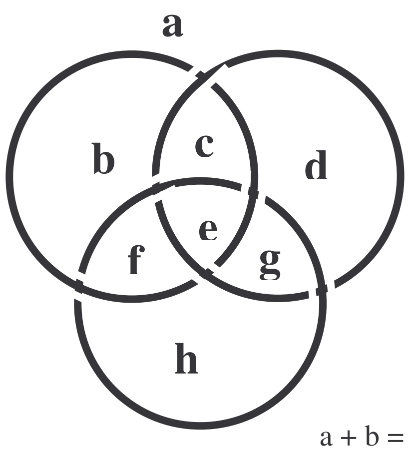
$$\text{F(figure 8 knot)} \cong \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & -1 & 1 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -2 & 1 & 1 \end{bmatrix} \cong \begin{bmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & 2 & 1 & 1 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & -1 & -2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 2 & 0 & -1 & 0 & -1 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 & -2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 2 & 3 & -2 & -3 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 3 & -2 & -3 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 5 & 0 & -5 \end{bmatrix} \cong [5, 0, -5] \cong [5, 0, 0]$$

$$\cong \mathbf{Z}_5 \oplus \mathbf{Z} \oplus \mathbf{Z}.$$

Since the face groups of the trefoil knot and the figure-8 knot are not isomorphic it follows that they're inequivalent knots.

Example 5: F(Borromean Rings)



$$F(\text{Borromean Rings}) = \left[a, b, c, d, e, f, g, h \;\middle|\; \left.\begin{array}{c} a+b=c+d \\ b+c=e+f \\ c+e=d+g \\ e+g=f+h \\ b+f=a+h \\ g+h=a+d \end{array}\right\} \right]$$

$$= \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 2 & -1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & -3 & -1 & 2 & 3 & 0 & -1 \\ 0 & -2 & -2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & -4 & 5 & 3 & -3 & -1 \\ 0 & -4 & 3 & 1 & -1 & 1 \end{bmatrix}$$

$$\cong \begin{bmatrix} 0 & 1 & -1 & 1 & -1 \\ -4 & 5 & 3 & -3 & -1 \\ -4 & 3 & 1 & -1 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 5 & -4 & 3 & -3 & -1 \\ 3 & -4 & 1 & -1 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 8 & -8 & 4 \\ 0 & -4 & 4 & -4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 8 & -8 & 4 \\ -4 & 4 & -4 & 4 \end{bmatrix} \cong \begin{bmatrix} -4 & 8 & -8 & 4 \\ 0 & -4 & 4 & 0 \end{bmatrix} \cong \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -4 & 4 & 0 \end{bmatrix} \cong \mathbf{Z}_4 \oplus [-4, 4, 0] \cong \mathbf{Z}_4 \oplus [4, 0, 0]$$

$$\cong \mathbf{Z}_4 \oplus \mathbf{Z}_4 \oplus \mathbf{Z} \oplus \mathbf{Z}.$$

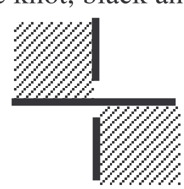
You'll have noticed by now that all the face groups we've obtained include two copies of **Z**. It's only the finite part that distinguishes them. In fact every face group includes at least two copies of **Z**.

10.3 The Alexander Group of a Link

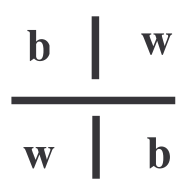
Theorem 2: The face group F(L) of a link L has the form F(L) ≅ **Z** ⊕ **Z** ⊕ A(L) for some abelian group A(L).

Proof: Being finitely generated F(L) ≅ **Z** ⊕ **Z** ⊕ … ⊕ **Z** ⊕ T where T is finite and where there are n copies of **Z**. We must show that $n \geq 2$.

Suppose you 2-colour a map of the knot, black and white.



Now add additional relations which equate all the generators for the black faces to b and all the white faces to w. The relations will all collapse to b + w = w + b.



Hence the group will become [b, w | b + w = w + b] ≅ [b, w |], with an empty set of relations. This is **Z** ⊕ **Z**. Hence $n \geq 2$.

The **Alexander Group** of a link L is the group A(L) such that the face group of L is isomorphic to **Z** ⊕ **Z** ⊕ A(L).

Example 6: A(trefoil) ≅ $\mathbf{Z}_3$, A(Borromean Rings) ≅ $\mathbf{Z}_4 \oplus \mathbf{Z}_4$, A(figure 8 knot) ≅ $\mathbf{Z}_5$.

For a link the Alexander Group can be infinite or finite, and if finite it can be either odd or even. But for a knot the Alexander Group must be finite and of odd order.

Theorem 3: The Alexander Group of a knot K is finite of odd order.

Proof: By suitably swapping overpasses and underpasses at certain crossings, any knot can be transformed to the unknot.

For example by changing one of the crossings in the figure 8 knot we get an unknot.

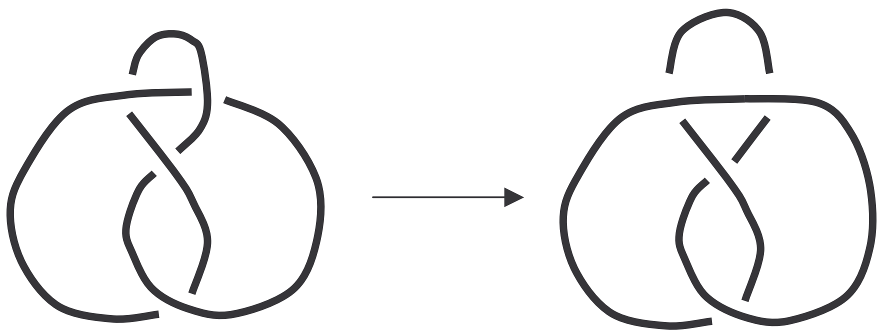


figure 8 knot **unknot**

The effect on the face group is to change some of the relations of the form a + b = c + d into a + c = b + d.

This makes no difference modulo 2. The face group of the unknot is clearly $\mathbf{Z} \oplus \mathbf{Z}$, which modulo 2 becomes $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Hence if $F(K) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \ldots \oplus \mathbf{Z} \oplus T$ where there are $(n \geq 2)$ copies of $\mathbf{Z}$ and where T is finite, we must have n = 2 and |T| odd.

The Alexander Group of a link is a useful invariant. However there is a fair amount of work in computing it. We can speed things up in two ways:

(1) Put the generators for two adjacent faces to zero. This has the effect of removing the two superfluous copies of Z and just gives the Alexander Group.

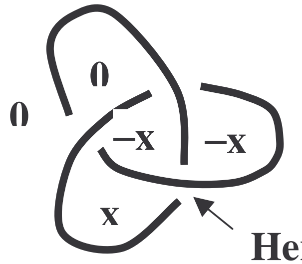
(2) Rather than label all the remaining faces with distinct labels and then writing down all the relations, simply use the generator "x" for any one of the remaining faces (it makes sense to put this adjacent to the other two 0's) and compute the remaining ones (as far as it is possible) in terms of x. While ever you have labels for 3 out of the 4 faces at a crossing you can use the relation for that crossing to label the 4th face.

In reasonably simple cases one can express every face in terms of this one generator "x". At the end there will be just one relation, of the form mx = 0, for some positive integer n, and then one can deduce that the Alexander group of the link is $\mathbf{Z}_m$.

In more complicated cases you may get to a point where, having labelled two adjacent faces as 0 and one face as "x" you get to a point where some, but not all, the faces can be expressed in terms of "x". If you reach this situation where you have no crossing with 3 out of the 4 faces labelled, then you simply label one of them "y". Then continue, expressing faces in terms of "x" and "y". In principle, with a very complicated link, you might need to introduce many new variables in this way. You'll end up with as many relations as variables.

If you introduce n variables you will have n relations connecting them and an n×n matrix of coefficients. If n > 1 there will be a little work in bringing it to diagonal form — but nowhere near as much work as if you had labelled the faces independently and hadn't removed the superfluous copies of **Z** by using two zeros.

Example 4:

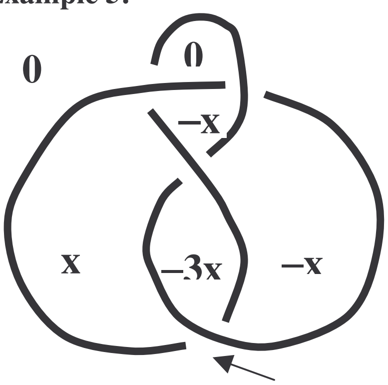


$$A(K) \cong [x \mid 3x = 0]$$
$$\cong \mathbf{Z}_3.$$

Here $x + 0 = -2x$

With a knot, where the Alexander group is finite, the **Alexander number** is defined to be the order of the Alexander group. This, of course, is also a knot invariant. However it's a little less discriminating than the Alexander group. For example, one knot has $\mathbf{Z}_9$ as its Alexander group and another knot has $\mathbf{Z}_3 \oplus \mathbf{Z}_3$ but both have Alexander number 9. The Alexander number can't tell them apart but the Alexander group can.

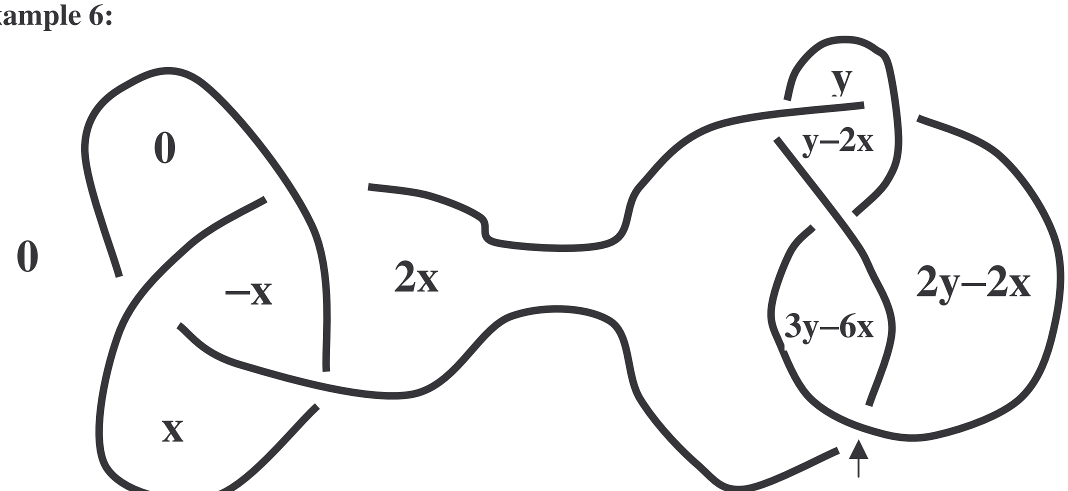
Example 5:



$$A(K) \cong [x \mid 5x = 0]$$
$$\cong \mathbf{Z}_5.$$

Here $x + 0 = -3x - x$

Example 6:



Here $2x = 5y - 8x$

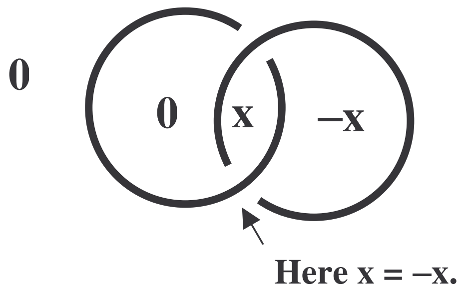
$A(K) \cong [x, y \mid 3x = 0, 2x = 5y - 8x] \cong [x, y \mid 3x = 0, 5y = 0] \cong \mathbf{Z}_3 \oplus \mathbf{Z}_5 \cong \mathbf{Z}_{15}$.

Example 7:

A(L) ≅ [x |] ≅ Z.

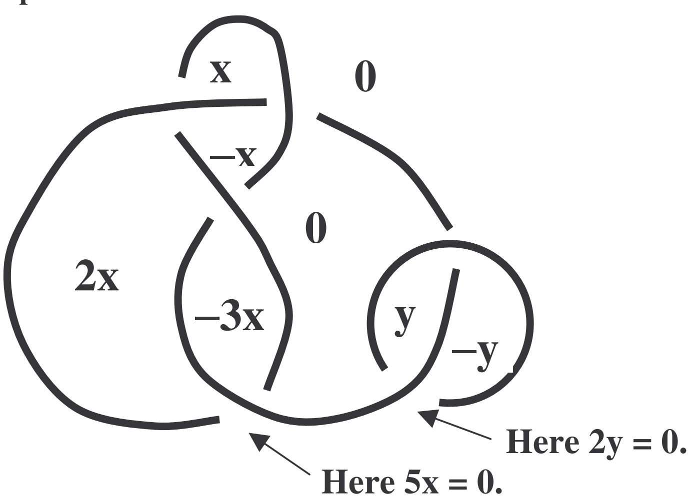
Since there are no crossings in this link, there are no relations. This shows that the Alexander group of a link can be infinite.

Example 8:



A(L) ≅ [x | 2x = 0] ≅ Z_2.
This shows that the Alexander group of a link can be finite of even order.

Example 9:



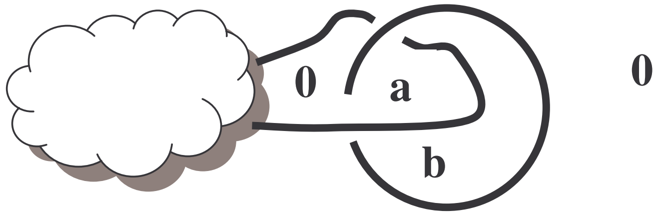
A(L) ≅ [x, y | 5x = 0, 2y = 0] ≅ $Z_5 \oplus Z_2 \cong Z_{10}$.

Theorem 4: If the link L′ is obtained from the link L by adding a linked ring then

A(L′) ≅ A(L) ⊕ Z_2.

Proof: Let $A(L) = [\dots \mid \dots]$.

Then L' has two additional faces:



$A(L') = [\dots , a, b \mid \dots , a + b = 0, a = b]$
$\cong [\dots , a \mid \dots , 2a = 0)]$
$\cong [\dots \mid \dots] \oplus [a \mid 2a = 0]$
$\cong A(L) \oplus \mathbf{Z}_2$.

Theorem 5: If the link L' is obtained from the link L by adding a disjoint ring then

$$A(L') \cong A(L) \oplus \mathbf{Z}.$$

Proof: Adding the disjoint ring adds one extra generator which is not involved in any relation.

Example 10:

n unlinked rings

$A(L) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \dots\dots \oplus \mathbf{Z}$ (with $n - 1$ copies of $\mathbf{Z}$).

Example 11: (Olympic Rings)

$A(L) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Example 12:

$A(L) \cong \mathbf{Z}_3 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

10.4 The Alexander Group of a Chain

A "chain", in the normal sense of the word, is a link in which each component is an unknot. Moreover these components are linked one to another in a particularly simple way.

The usual word for each component is "link". We talk about the chain being made up of separate "links". But here the word "link" has a different meaning from the technical one, where we would call the whole chain a link. To avoid this confusion we'll use everyday language in this context. We'll refer to the whole object of study as a "chain", rather than a "link" and each separate component as a "link", rather than a "component". So the above object is a chain with 8 links.

As a corollary to Theorem 4 we have the fact that the Alexander group of a chain with n links is $\mathbf{Z}_2{}^{n-1}$, the direct sum of $n - 1$ copies of $\mathbf{Z}_2$.

Now there are two ways that a pair of adjacent links can occur in a projection of a chain and we shall refer to these as positive and negative connections as follows.

+ve connection

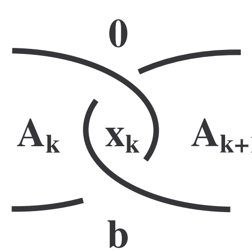
–ve connection

It is obvious that this distinction only occurs at the level of projections because both are equivalent for a chain. One link can be rotated relative to the other to achieve the other projection and so the Alexander group of a chain is independent of positive and negative connections. That is, until the ends of the chain are joined. (We call this a **closed chain**.)

The following theorem was obtained by a Macquarie University student in 2004 while on a vacation scholarship.

Theorem 5 (Simon Byrne): Suppose a chain with $n \geq 2$ links has its ends joined. Let m be the absolute difference between the number of positive and negative connections in some projection in which the only crossings are from one link to the next.
The Alexander group of this closed chain is $\mathbf{Z}_2{}^{n-2} \oplus \mathbf{Z}_{2m}$ if $m > 0$ and $\mathbf{Z}_2{}^{n-2} \oplus \mathbf{Z}$ if $m = 0$.
Proof: Suppose we have a chain with n links. Make a projection in which the chain, ignoring the fact that it is made up of links, does not cross itself. That is, the only crossings are those that arise from links connecting adjacent ones. There are $2n + 2$ faces. One is the region outside the chain, which we shall label as 0. Another is the inside of the chain, which we shall label as b. Then there are n faces, being the regions surrounded by pieces of adjacent links. We shall label these $x_1, x_2, \ldots, x_n$. Finally there are n faces that form the centre part of the links which we shall label as $A_1, A_2, \ldots, A_n$.

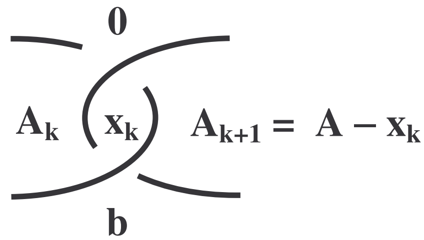
For a positive connection we have:



$$A_{k+1} = A_k + x_k$$

Here we have the relation $A_k + b = A + 2x_k$ which simplifies to $2x_k = b$.

For a negative connection we have:



Here we have the relation $A_k + x_k = A - x_k + b$ which again simplifies to $2x_k = b$.

So far we have only set the generator for one face equal to 0. We now set $A_1 = 0$. Then for each k

$A_{k+1} = \sum_{i=1}^{k} c_i x_i$ where $c_i = 1$ if x_i corresponds to a positive connection and -1 if it corresponds to a negative one. But because the n'th link connects to the first this will give $\sum_{i=1}^{n} c_i x_i = A_1 = 0$.

The Alexander group is $[x_1, x_2, \ldots, x_n, b \mid \sum_{i=1}^{n} c_i x_i = 0, 2x_1 = b, 2x_2 = b, \ldots 2x_n = b]$

Multiplying the first relation by 2 and using the remaining ones we deduce the additional relation $\left(\sum_{i=1}^{n} c_i\right) b = 0$.

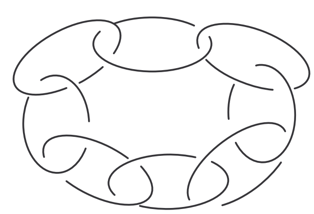
Hence the Alexander group is $[x_1, x_2, \ldots, x_{n-1}, b \mid mb = 0, 2x_1 = b, 2x_2 = b, \ldots 2x_{n-1} = b]$ where $m = \mathrm{ABS}\left(\sum_{i=1}^{n} c_i\right)$.

Writing $y_i = x_i - x_1$, for $i = 2, \ldots, n$ we can write this as:

$[x_1, y_2, \ldots, y_{n-1}, b \mid mb = 0, 2x_1 = b, 2y_2 = 0, \ldots 2y_{n-1} = 0]$

$\cong [x_1, x_2, \ldots, x_{n-1} \mid 2mx_1 = 0, 2y_2 = 0, \ldots 2y_{n-1} = 0] \cong \mathbf{Z}_2^{\,n-2} \oplus \mathbf{Z}_{2m}$ if $m > 0$ and $\mathbf{Z}_2^{\,n-2} \oplus \mathbf{Z}$ if $m = 0$.

Example 13: Find the Alexander group of the following closed chain.



Solution: There are 8 links and so 8 connections, of which 2 are positive and 6 are negative. Using the terminology of Theorem 5, n = 8 and m = 6 − 2 = 4. Hence the Alexander group is $\mathbf{Z}_2^{\,6} \oplus \mathbf{Z}_8$.