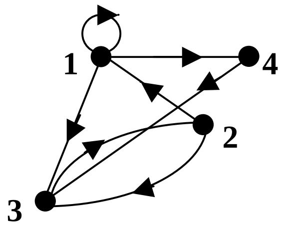
5. GRAPHS ON SURFACES

5.1 Graphs

A graph, by itself, is a combinatorial object rather than a topological one. But when we relate a graph through the process of "embedding" we move into the realm of topology.

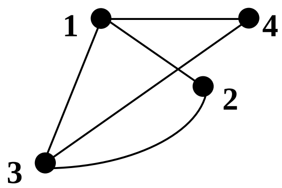
A **graph** is a set, X, of elements together with a relation on X. We often draw pictures of graphs, where the elements of X are represented by dots and the relation by a set of arrows connecting certain dots to others. The elements of X are called **vertices** and the arrows joining vertices related by the relation are called **edges**.

Example 1: The following is a graph on the set {1, 2, 3, 4}:
{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (3, 2), (4, 3)}. This can be represented by the following diagram. There are 4 vertices and 7 edges.



We shall only consider "undirected graphs with no loops". A **loop** is an edge (x, x) from a vertex to itself. An **undirected graph** is one where the relation is symmetric – if x is related to y then y is related to x. So if there's an arrow in one direction there's always one in the opposite direction. In an undirected graph there's no need to use arrows to connect vertices since we know that the relation goes in both directions. So we simply use edges without arrows.

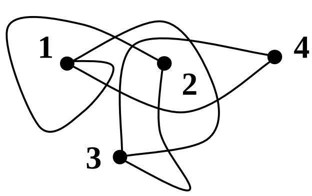
Example 2: The following is an undirected graph with no loops.

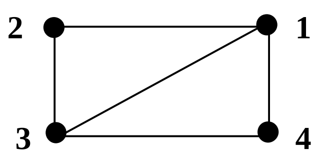


From now on, when we use the word "graph", we'll mean that it's undirected and has no loops.

A graph is a combinatorial structure where the only consideration is which vertices are adjacent to which. When we draw a graph the positions of the points representing the vertices are arbitrary. So are the routes of the edges. The edges needn't be straight, they're allowed to cross over other edges, and they could even wind around in more complicated ways. However we usually draw a graph in such a way that it gives as simple a picture as possible.

Example 3: The graph in example 2 could be redrawn as in the diagram on the left, but would look much better when drawn as the one on the right.





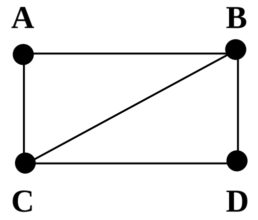
Notice that in the above example it is possible to draw the graph without any of the edges crossing. This isn't always possible, and we'll be very much concerned with the problem of when this is possible and when it isn't.

In a graph we say that two vertices are **adjacent** if there's an edge between them.

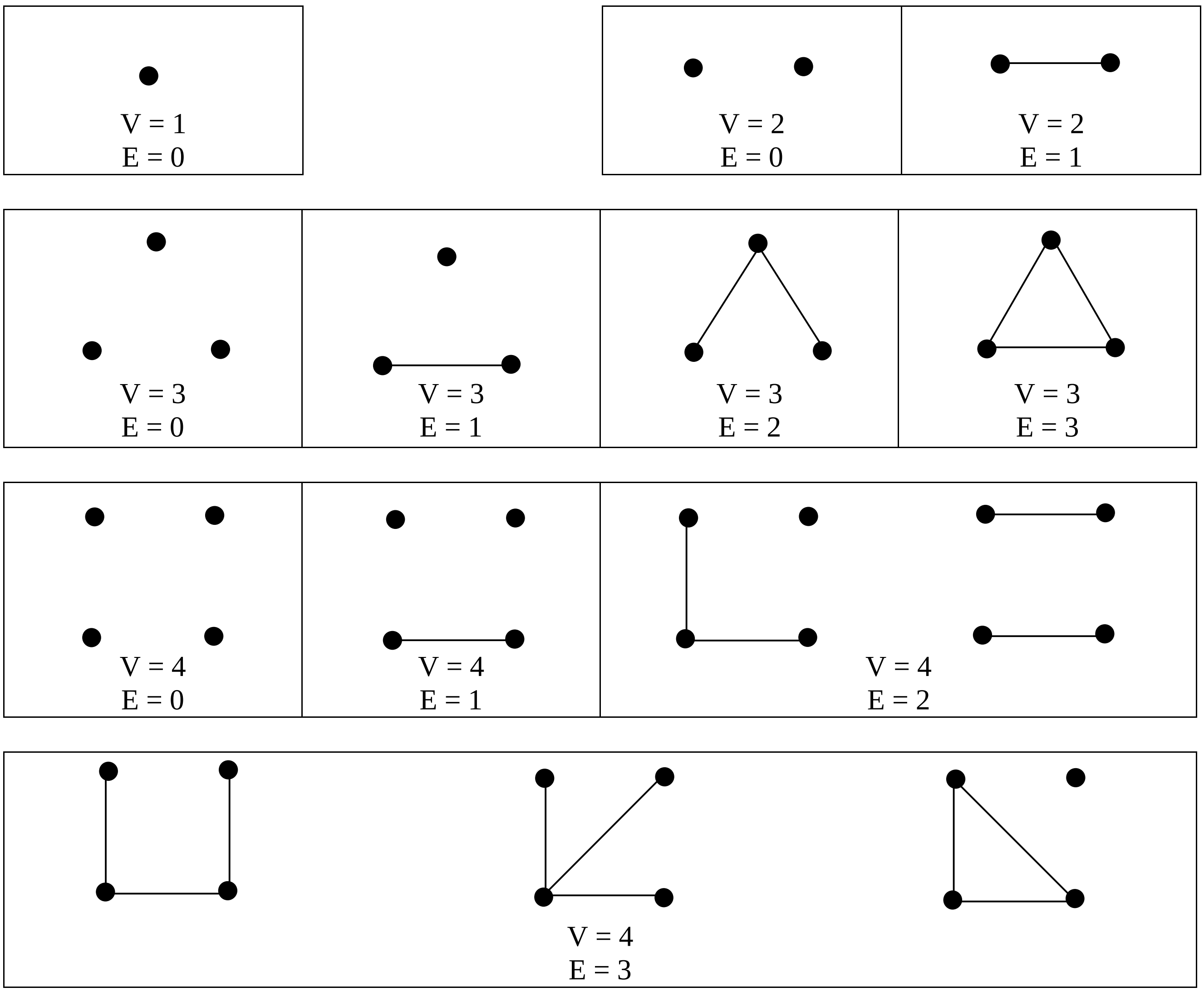
Example 4: In example 3, vertices 1 and 2 are adjacent but 2 and 4 are not.

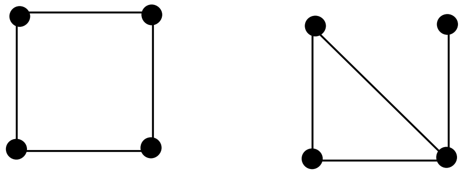
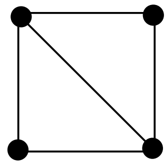
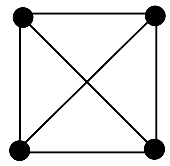
Two graphs X and Y are **equivalent** if there's a 1-1 and onto map $f: X \to Y$ such that x_1 and x_2 are adjacent in X if and only if $f(x_1)$ and $f(x_2)$ are adjacent in Y.

Example 5: The two graphs in example 3 are equivalent. And both are equivalent to the following graph. (It's essentially the same graph but with the vertices labelled differently.)



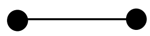
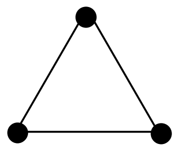
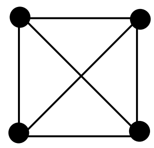
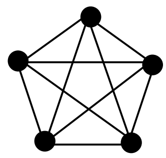
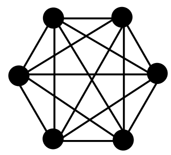
Example 6: The following list contains all the graphs with 4 vertices or less. They have been systematically classified according to the number of vertices, V and edges, E.



 V = 4 E = 4	 V = 4 E = 5	 V = 4 E = 6

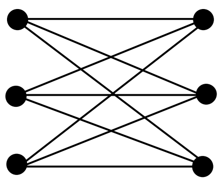
The **complete graph** on n vertices, denoted by $\mathbf{K_n}$, is the graph where every vertex is adjacent to every other vertex. The number of edges in a complete graph on n vertices is clearly the binomial coefficient $\binom{n}{2}$.

Example 7: The following are the complete graphs on 6 vertices or less.

$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	$\mathbf{K_5}$	$\mathbf{K_6}$
					

Another important family of graphs consists of the graphs $\mathbf{K_{m,n}}$ for various values of m and n (they don't have a name, just a symbol). The graph $\mathbf{K_{m,n}}$ has m + n vertices divided into two subsets, one of size m and the other of size n. Every vertex in one subset is adjacent to every vertex in the other, but there are no edges connecting two vertices within the same subset.

Example 8: The following is $\mathbf{K_{3,3}}$:



This graph was once featured in an Air New Zealand advertisement, where the 6 vertices consisted of the cities Brisbane, Sydney, Melbourne, Auckland, Wellington and Christchurch. The edges represented the trans-Tasman routes.

5.2 Embedding a Graph in a Surface

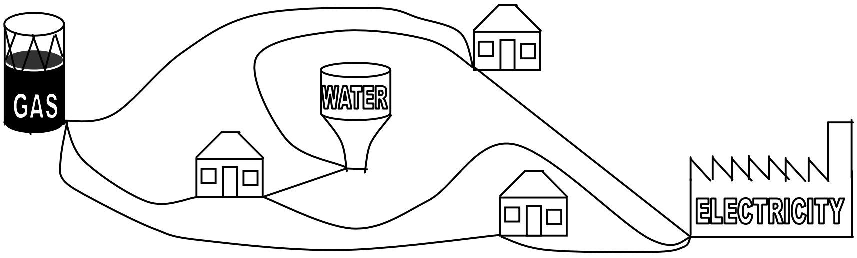
A graph has vertices and edges. So has a map, so what's the difference? Maps have faces, while graphs don't. It's the existence of faces on a map that give it its topological significance. Consider the following famous puzzle, called the Utilities Puzzle.

You have three houses and three "utilities". The utilities are a gasworks, a power station and a water reservoir. They have to pump gas, electricity and water to each of the three houses. But they have to do this so that the pipes and cables don't intersect.

You see this is a 2-dimensional problem. In real life (3-dimensional) there's no problem at all. Gas pipes can be routed over electrical cables or under water pipes. But we have to solve the puzzle in 2 dimensions.

The problem is to draw the three houses and three utilities on a sheet of paper, and draw lines to represent the pipes and cables, in such a way that they only meet one another at endpoints. You might like to have a go at this problem.

Example 9: The following is a "near" solution. Clearly we can't put in the remaining water pipe without crossing the lines we've drawn already.



But that doesn't prove that the problem has no solution. Perhaps you can think of better places to put the six vertices or perhaps you can route the first 8 edges in a really clever way so that the last one can be drawn in without crossing the others.

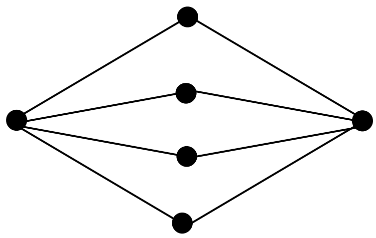
Don't spend too long on this puzzle, because it's impossible! If you've tried to solve it for a few minutes you'll come to this conclusion, though you won't have a proof. But beware! Haven't you ever attempted puzzles where, after trying in vain for many minutes, you become quite convinced that it's impossible, only to have someone come along and show you a really clever solution? Not in this case, though. We're going to *prove* that this puzzle is impossible!

In the language of graph theory the network of pipes and cables is the graph $\mathbf{K_{3,3}}$. We want to draw this graph in the plane so that edges meet only at vertices. Or, to use a new concept that we're just about to define, the problem is to *embed* $\mathbf{K_{3,3}}$ on a disk.

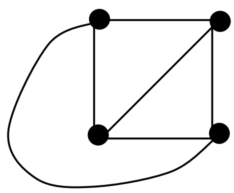
A graph can be **embedded** (is **embeddible**) on a surface if it can be drawn on that surface so that edges meet only at vertices. So we'll be proving that $\mathbf{K_{3,3}}$ is not embeddable on a disk.

5.3 Planarity

A graph is defined to be **planar** if it can be embedded in a disk. So $\mathbf{K_{3,3}}$ isn't planar. But $\mathbf{K_{4,2}}$ is:



So is $\mathbf{K_4}$:



Since a disk can be cut out of a sphere, any planar graph can be embedded in a sphere. On the other hand, if we can embed a graph in a sphere, we can cut out a small hole in the middle of one of the faces and we have an embedding of the graph in a disk. (Remember that a disk is homeomorphic to a sphere with one hole.)

So a graph is planar if and only if it can be embedded in a sphere. This is useful because often a sphere is more convenient to work with.

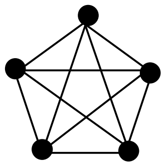
To prove that a graph is planar we can simply draw it, with edges not crossing. But how do we show that a graph, such as, $\mathbf{K_{3,3}}$ or $\mathbf{K_5}$ isn't planar? The technique discussed here is to work out the average number of edges per face and compare this to the smallest number of edges per face. But wait a minute. Graphs don't have faces!

That's true, but a graph embedded in a surface becomes a map, and maps have faces. So we *suppose* that the graph is planar, that is, that it can be embedded in a sphere. But how can we count the number of faces if we're only *supposing* that the graph can be embedded? The answer is to use Euler's formula:

$$V + F - E = \chi$$

For planarity we use $\chi = 2$, the Euler characteristic of the sphere. Why not $\chi = 1$ for the disk? The answer is that we'll be assuming that there's a face on both sides of each edge of the map. If we have boundaries this will not be so.

Example 10: $\mathbf{K_5}$ is not planar.



Proof: For $\mathbf{K_5}$ we have V = 5 and E = 10.
Suppose that $\mathbf{K_5}$ is planar. Then embedding it in a sphere we can deduce that the number of faces must be:

$$F = 2 + E - V = 7.$$

The average number of edges per face must therefore be $\frac{2E}{F} = \frac{20}{7} = 2\frac{6}{7}$.

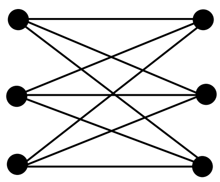
Why $\frac{2E}{F}$ and not just $\frac{E}{F}$? The reason is that every edge is associated with two faces – one on each side. So if you were to split each edge lengthwise, so that each half edge was associated with only one face, we'd have 2E half edges to share among the F faces.

Now we wanted to prove that $\mathbf{K_5}$ can't be embedded in a sphere and we started out by supposing that it can be. We're clearly looking for a contradiction. So what's contradictory about the average number of edges per face being $2\frac{6}{7}$?

What's wrong is that it's less than 3. Every face must be surrounded by at least 3 edges (A face bounded by 2 edges would require that the two edges connect the same two vertices, and a face bounded by just 1 edge would mean that the graph has a loop.)

Now the average of a collection of numbers can't be less than the smallest of them. So here we have our contradiction!

Example 11: $\mathbf{K_{3,3}}$ is not planar.



Proof: Here V = 6 and E = 9.

Suppose that $\mathbf{K_{3,3}}$ can be embedded in a sphere. The resulting map would have to have F faces where 6 + F – 9 = 2, that is it must have 5 faces.

The average number of edges per face would therefore be $\frac{18}{5} = 3\frac{3}{5}$.

This isn't less than 3, so where's the contradiction? The contradiction is that it's less than 4. You see, in this graph there are no cycles of length 3. Each edge takes you from one set of vertices to the other. Going along another edge must take you back to a different vertex in the first set. The smallest cycles have length 4. The boundary of a face must be a cycle in the graph. So the smallest number of edges for each face is 4. But the average of these numbers is less than 4. This can't be, and so we have our contradiction.

The **girth** of a graph is the length of the shortest cycle. The girth of $\mathbf{K_5}$ is 3 but the girth of $\mathbf{K_{3,3}}$ is 4. We get a contradiction if the average number of edges per face is less than the girth.

5.4 Proving that a Graph is not Embeddible in a Surface.

Suppose a graph has E edges and V vertices and the girth is g. If the graph can be embedded in a surface with Euler characteristic χ then it will produce a map with F faces where F = 2 + E – V. If $\mathbf{\frac{2E}{F} < g}$ then we have a contradiction. That graph can't possibly be embedded in that surface.

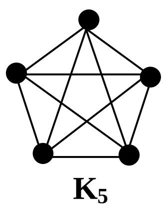
But beware. The $\frac{2E}{F} < g$ is a *one-way* test. If, on the contrary, we find that $\frac{2E}{F} \geq g$ we'd be wrong to conclude that the graph *is* embeddible. It may be embeddible, or on the other hand it may be non-embeddible after all and this test isn't powerful enough to show it.

This is one of the most common errors in graph embedding. Remember that essentially **the only way to prove that a graph IS embeddible is to actually draw an embedding**.

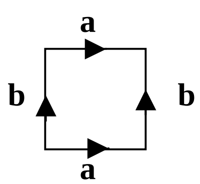
Test for Non-Embeddiblility only applies to surfaces with no boundaries.‡	
$\mathbf{F = \chi + E - V}$	
$\mathbf{\frac{2E}{F} < g}$	**A graph with E edges, V vertices and girth g is NOT embeddable in any surface with Euler characteristic χ**
$\mathbf{\frac{2E}{F} \geq g}$	**The graph may be embeddable. On the other hand it might not be. In this case the TEST FAILS.**

‡ If the surface has boundaries, remove them and use the corresponding surface with no boundaries.

Example 12: The graph $\mathbf{K_5}$ is embeddable in the torus.



K5



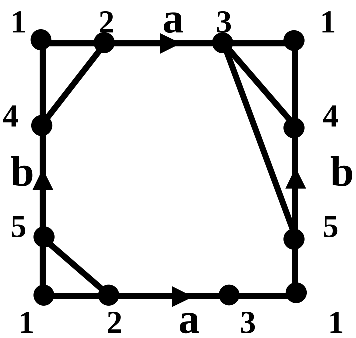
torus

Non-Proof: For $\mathbf{K_5}$, V = 5 and E = 10.
Suppose that $\mathbf{K_5}$ can be embedded in a torus. The resulting map will have F faces where:

$5 + F - 10 = 0$ (remember that $\chi = 0$ for the torus). So $F = 5$.

$\frac{2E}{F} = 4$. The girth of $\mathbf{K_5}$ is 3. So $\frac{2E}{F} > g$. What does this prove? Absolutely nothing! This whole calculation achieves nothing beyond confirming that the claimed result *might* be true.

Proof: To prove that $\mathbf{K_5}$ is embeddible in a torus we simply take a torus and draw $\mathbf{K_5}$ on it (in such a way that edges don't cross). Of course it's easier to use a polygon with identified edges rather than the surface of a real 3-dimensional torus.



Note that there are alternative routes we could have followed for the 3-5 connection. We should only include one of them. (The apparent double-up of the 2-3 connection is an illusion as the 2-3 edge at the top and the bottom of the square are, in reality, the same edge.)

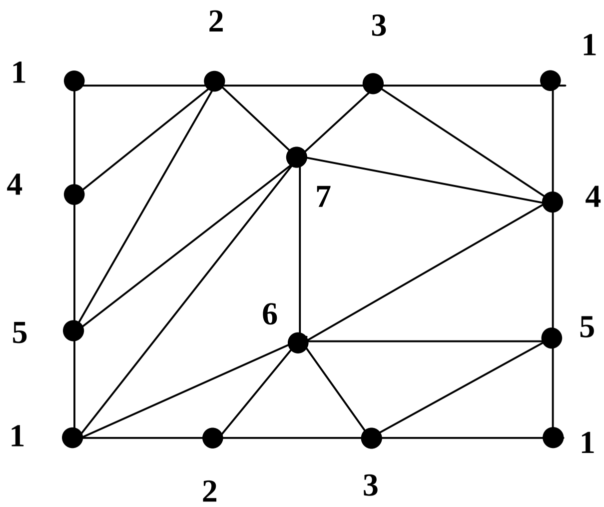
The duplication of the labels of the vertices comes about because of the identification of the edges. Check that every vertex, from 1 to 5, is adjacent to every other.

Example 13: The largest value of n for which $\mathbf{K_n}$ is embeddable in the torus is 7.
Solution: For $\mathbf{K_n}$, $V = n$ and $E = ½\, n(n - 1)$.
Suppose that $\mathbf{K_n}$ can be embedded in a torus. The resulting map will have F faces where:

$n + F - ½\, n(n - 1) = 0$. So $F = \frac{n^2 - 3n}{2}$.

$\frac{2E}{F} = \frac{2n(n-1)}{n^2 - 3n}$. The girth of $\mathbf{K_5}$ is 3. So $\frac{2n(n-1)}{n^2 - 3n} \geq 3$, in which case $3n^2 - 9n \leq 2n^2 - 2n$.
Hence $n^2 - 7n \leq 0$. Since $n > 0$ we may divide this inequality by n to obtain $n \leq 7$.

We now show that $\mathbf{K_7}$ can be embedded in a torus.



Suppose you have a graph G and a surface S an you want to determine whether or not G is embeddable in S. The first thing you might try is the $\frac{2E}{F} < g$ test. But before you do, there are a couple of things you should do first to both the graph and the surface:

(1) Clearly holes are irrelevant to the embedding problem. A graph is embeddable in a surface with holes if and only if it is embeddable in the corresponding surface with no holes. But the $\frac{2E}{F} < g$ test assumes that the surface has no holes. So the very first thing you must do is to:

replace the surface by the corresponding surface with no holes.

If you're asked whether a graph is embeddible in a disk, or a cylinder, you replace that surface by a sphere. In testing embeddability in a Möbius Band you replace it by a projective plane.

(2) If the graph has a vertex of degree 1 you can clearly remove that vertex, and the edge joined to it, without affecting embedability. Furthermore, you can remove a vertex of degree 2 and join the edges on either side into a single edge, without affecting embedability. However removing vertices of degree 2 may reduce the girth, which will weaken the test. So before you apply the $\frac{2E}{F} < g$ test you should:

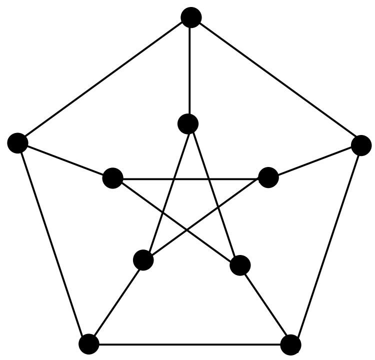
remove vertices of degrees 1 and
remove any vertices of degree 2 if so doing does not change the girth.

Having modified the graph and surface appropriately you can now apply the $\frac{2E}{F} < g$ test. If it holds, the graph is not embeddable in the surface. If $\frac{2E}{F} \geq g$ then the test fails, that is, it is inconclusive.

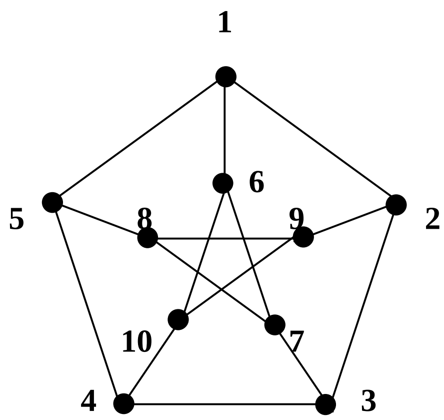
The next thing to do would be to attempt to draw an embedding.

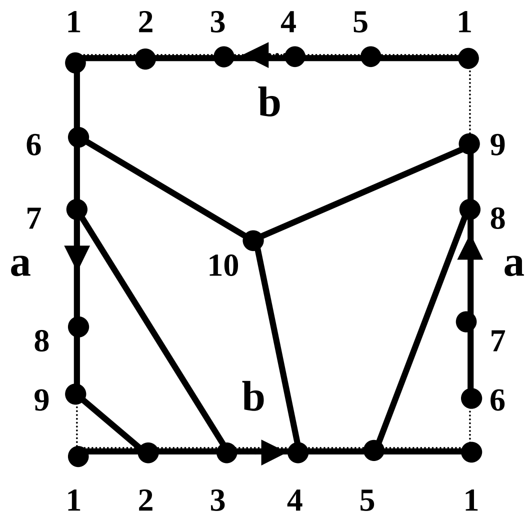
Example 14: The following graph is known as the Petersen Graph:

V = 10
E = 15
If embedded in a sphere
F = 2 + 15 – 10 = 7
$f = \frac{2E}{F} = \frac{30}{7} < 5$
g = 5
So the Petersen graph can't be embedded in a sphere.
In other words it isn't planar.



For a projective plane, $\chi = 1$ and so if this graph can be embedded in a PP, F = 1 + 15 – 10 = 6 and $f = \frac{2E}{F} = \frac{30}{6} = 5 = g$. The $\frac{2E}{F} < g$ test fails. So it might be possible to embed the Petersen graph in a projective plane. On the other hand it might be impossible, which isn't much help!
But in fact we *can* embed the Petersen graph in a projective plane by actually displaying such an embedding:





Whenever you display such an embedding it's important to do what we've done here – label the vertices of the original graph and the embedded graph so that it's easy to check that it is indeed the same graph. (Vertex 1 is adjacent to 2 in both graphs, it is not adjacent to vertex 3 in either graph, and so on.)

5.5 Printed Circuit Boards.

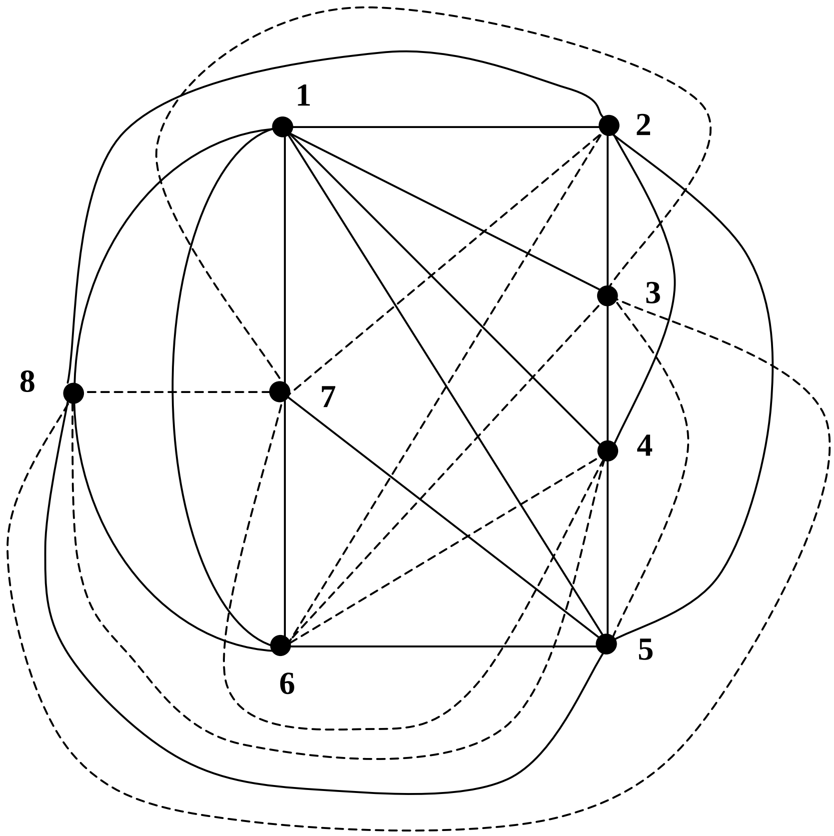
A printed circuit board has electronic components laid out on both sides of a board with the connecting tracks "printed" on the board. They can be considered as graphs where the vertices occur on both sides of the surface and the edges on each side form a planar graph. We insist that each edge lies entirely on one side or the other. (Of course any graph can be laid out on a printed circuit board by adding extra vertices of degree 2 and avoiding crossings by sending one of the edges briefly to the other side.)

A graph is **2-embeddable** in a 2-sided (orientable) surface S if it is the union of two subgraphs, each of which is embeddable in S.

Theorem: If K_n is 2-embeddable in a sum of m toruses then $n \le \dfrac{13 + \sqrt{73 + 96m}}{2}$.

Proof: Suppose the numbers of edges in the subgraphs are E_1 and E_2, where
$E_1 + E_2 = ½ n(n – 1)$. The Euler characteristic of the sum of m toruses is 2 – 2m.
Suppose that the numbers of faces n the corresponding maps are F_1 and F_2. For each subgraph we must have $\dfrac{2E_i}{F_i} \ge 3$, that is, $2E_i \ge 3F_i = 3(E_i + 2 - 2m - n)$.
Hence $E_i \le 3n + 6m - 6$ and so $½ n(n - 1) \le 6n + 12m - 12$ and so $n^2 - 13n - 24(m - 1) \le 0$.
The roots of $^2 - 13n - 24(m - 1)$ are $\dfrac{13 \pm \sqrt{169 + 96(m - 1)}}{2}$ = and so
$n \le \dfrac{13 + \sqrt{73 + 96m}}{2}$.

Example: The largest value of n for which K_n can be 2-embedded in a plane is 8, 9 or 10. If K_n is 2-embeddable in a plane then from the above theorem $n \le 10$.
The following is a 2-embedding of K_8 in a plane.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