

3. PAPPUS' THEOREM

§3.1. Duality

If P is a projective point (1-dimensional subspace) then P^\perp is a projective line (2-dimensional subspace) and if h is a projective line then h^\perp is a projective point. The relation $P \leftrightarrow P^\perp$ (or equivalently $h \leftrightarrow h^\perp$) establishes a 1-1 correspondence between the points and lines of the real projective plane. Moreover it interacts with the incidence structure in a very nice way.

Theorem 6: The projective point P lies on the projective line h if and only if the projective point h^\perp lies on the projective line P^\perp .

Proof: Let $P = \langle \mathbf{p} \rangle$ and let $h = \langle \mathbf{v} \rangle^\perp$. Then P lying on h is equivalent to \mathbf{p} and \mathbf{v} being orthogonal. But $h^\perp = \langle \mathbf{v} \rangle^{\perp\perp} = \langle \mathbf{v} \rangle$ and so h^\perp lying on P^\perp is also equivalent to \mathbf{p} and \mathbf{v} being orthogonal. So the two geometric statements are equivalent to one another.

As a consequence of this very innocent-looking theorem we can establish the following very powerful Principle of Duality.

The Principle of Duality

Any theorem in projective geometry that can be expressed in terms of the following six concepts remains true if these concepts are interchanged as follows:

projective point \leftrightarrow *projective line*
lies on \leftrightarrow *passes through*
collinear \leftrightarrow *concurrent*

A concept or theorem that is obtained by the above interchanges is called the dual of the original one. What this principle means is that every time we prove a theorem in projective geometry we automatically have proved another theorem, its dual. Well that is not quite true because sometimes a theorem is its own dual. But in the case of Desargues' the dual of what we proved is the converse.

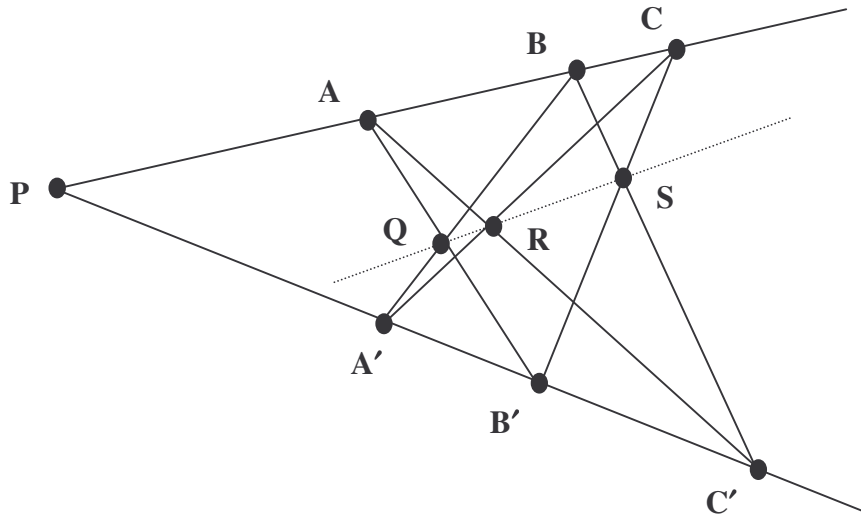
The dual of the property of two triangles being in perspective from a point is two triangles being in perspective from a line. (It's not just because we have changed the word point to line. You need to look at their definitions to see that the definitions are duals of one another.) We proved that if triangles are in perspective from a point then they are in perspective from a line. The dual, which must be true by the principle of duality, is that if triangles are in perspective from a line then they are in perspective from a point.

Sometimes, as in Desargues' theorem, the dual turns out to be the converse. Sometimes it turns out to be the same theorem (a self-dual theorem). Sometimes it is a totally different theorem. This is the case with the next theorem.

§3.2. Pappus' Theorem

Pappus of Alexandria (c. 300 A.D.) was a Greek mathematician who provided a particularly simple proof of the equality of the base angles of an isosceles triangle. His great work "A Mathematical Collection" is an important source of information about ancient Greek mathematics.

Theorem 7: Suppose $\{A, B, C\}$ and $\{A', B', C'\}$ are two collinear sets where the 6 points are distinct and the two lines are distinct.



Let $Q = AB' \cap A'B$, $R = AC' \cap A'C$ and $S = BC' \cap B'C$. Then Q, R, S are collinear.

Proof: Let $P = AB \cap A'B'$. The theorem is trivial if P coincides with any of the six points so we may assume that it is distinct from each of them.

By the Collinearity Lemma we may choose vectors \mathbf{a}, \mathbf{a}' and scalars λ, λ' such that:

$$\begin{aligned} P &= \langle \mathbf{p} \rangle; \\ A &= \langle \mathbf{a} \rangle, A' = \langle \mathbf{a}' \rangle; \\ B &= \langle \mathbf{p} + \mathbf{a} \rangle, B' = \langle \mathbf{p} + \mathbf{a}' \rangle; \\ C &= \langle \lambda \mathbf{p} + \mathbf{a} \rangle, C' = \langle \lambda' \mathbf{p} + \mathbf{a}' \rangle. \end{aligned}$$

Let $\mathbf{q} = (\mathbf{p} + \mathbf{a}) + \mathbf{a}' = (\mathbf{p} + \mathbf{a}') + \mathbf{a} \in (A' + B) \cap (A + B')$. Hence $Q = \langle \mathbf{q} \rangle$.

Let $\mathbf{r} = \lambda \lambda' \mathbf{p} + \lambda' \mathbf{a} + \lambda \mathbf{a}' = \lambda(\lambda' \mathbf{p} + \mathbf{a}) + \lambda' \mathbf{a} = \lambda'(\lambda \mathbf{p} + \mathbf{a}) + \lambda \mathbf{a}' \in (A + C') \cap (A' + C)$.

Hence $R = \langle \mathbf{r} \rangle$.

$$\begin{aligned} \text{Finally let } \mathbf{s} &= \mathbf{q} - \mathbf{r} = \mathbf{p} + \mathbf{a} + \mathbf{a}' - \lambda \lambda' \mathbf{p} - \lambda' \mathbf{a} - \lambda \mathbf{a}' \\ &= (1 - \lambda \lambda')(\mathbf{p} + \mathbf{a}) + (1 - \lambda)(\lambda' \mathbf{p} + \mathbf{a}') \in B + C' \\ &= (1 - \lambda)(\mathbf{p} + \mathbf{a}') + (1 - \lambda \lambda')(\lambda \mathbf{p} + \mathbf{a}) \in B' + C. \end{aligned}$$

Hence $S = \langle \mathbf{s} \rangle$.

Since $\mathbf{q} = \mathbf{r} + \mathbf{s}$ it follows that $\mathbf{q}, \mathbf{r}, \mathbf{s}$ are linearly dependent and so Q, R, S are collinear.

§3.3 Finite Projective Planes

So far we have only discussed the real projective plane. We took the field of real numbers \mathbf{R} and formed the 3-dimensional vector space over it, \mathbf{R}^3 . Our projective points were 1-dimensional subspaces of \mathbf{R}^3 and our lines were 2-dimensional subspaces.

Everything we have done so far would have worked if we had taken any field F and worked within the 3-dimensional vector space F^3 . Our scalars would have been the elements of F . Every piece of algebra we carried out would have been valid for any such field. (Note that there was one place where we needed to assume that the scalars commute under multiplication, which of course is valid in any field. Can you find where we invoked the commutative law for multiplication?)

We denote the projective plane that is formed from the field F by the symbol $\wp(F)$. So the real projective plane is $\wp(\mathbf{R})$.

We could take, instead of the field of real numbers, the field of complex numbers and so produce the complex projective plane $\wp(\mathbf{C})$. Or we could have manufactured the rational projective plane $\wp(\mathbf{Q})$. All our theorems, including Desargues' and Pappus' theorems would remain valid. (We may have had difficulty in drawing diagrams in the complex projective plane but the diagrams are not essential to the proofs.)

We could also take our field to be finite and so produce finite projective planes. Again diagrams would be difficult to draw but again there would be no need. In a finite projective plane, projective lines have finitely many points so we need only list the points to completely describe the line.

The simplest finite fields are the integers modulo a prime. The field of integers modulo p is denoted by \mathbf{Z}_p . The smallest field is \mathbf{Z}_2 , the field with 2 elements. These elements are written 0, 1 and addition and multiplication can be described by the tables:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Let's examine the smallest projective plane, $\wp(\mathbf{Z}_2)$. The field \mathbf{Z}_2 has 2 elements. The vector space \mathbf{Z}_2^3 has $2^3 = 8$ elements. Of these one, the zero vector $(0, 0, 0)$ doesn't span a projective point. The other 7 do. Moreover, because there is only one non-zero scalar, each of these 7 vectors spans a different projective point. So $\wp(\mathbf{Z}_2)$ has 7 points:

$$\langle(0, 0, 1)\rangle, \langle(0, 1, 0)\rangle, \langle(0, 1, 1)\rangle, \langle(1, 0, 0)\rangle, \langle(1, 0, 1)\rangle, \langle(1, 1, 0)\rangle, \langle(1, 1, 1)\rangle.$$

By duality there must be the same number of projective lines:

$$\langle(0, 0, 1)\rangle^\perp, \langle(0, 1, 0)\rangle^\perp, \langle(0, 1, 1)\rangle^\perp, \langle(1, 0, 0)\rangle^\perp, \langle(1, 0, 1)\rangle^\perp, \langle(1, 1, 0)\rangle^\perp, \langle(1, 1, 1)\rangle^\perp.$$

For example the projective point $\langle(0, 1, 1)\rangle$ lies on the projective line $\langle(1, 1, 1)\rangle^\perp$ since

$$(0, 1, 1) \cdot (1, 1, 1) = 0 + 1 + 1 = 0.$$

How many points are there on each of these 7 lines? A projective line is a 2-dimensional subspace and so has $2^2 = 4$ vectors. Of these the zero vector doesn't span a projective point but the other 3 do. They span 3 distinct projective points. So on each line in $\wp(\mathbf{Z}_2)$ we have 3 points. And how many lines through each point? That's easy by duality – the same number – three.

Now it's very messy working with these vectors and having to do a modulo 2 dot product every time we want to see if a point lies on a line. It is much easier to code them and to construct a Collinearity Table.

We shall code the 7 points as A, B, C, D, E, F, G and the 7 lines as a, b, c, d, e, f, g.

$$\begin{array}{ll}
 A = \langle(0, 0, 1)\rangle, & a = \langle(0, 0, 1)\rangle^\perp; \\
 B = \langle(0, 1, 0)\rangle, & b = \langle(0, 1, 0)\rangle^\perp; \\
 C = \langle(0, 1, 1)\rangle, & c = \langle(0, 1, 1)\rangle^\perp; \\
 D = \langle(1, 0, 0)\rangle, & d = \langle(1, 0, 0)\rangle^\perp; \\
 E = \langle(1, 0, 1)\rangle, & e = \langle(1, 0, 1)\rangle^\perp; \\
 F = \langle(1, 1, 0)\rangle, & f = \langle(1, 1, 0)\rangle^\perp; \\
 G = \langle(1, 1, 1)\rangle. & g = \langle(1, 1, 1)\rangle^\perp.
 \end{array}$$

Now, with a certain amount of calculation we can work out which points lie on which line and so produce the following Collinearity Table.

	a	b	c	d	e	f	g
B	A	C	A	B	A	C	
D	D	D	B	E	F	E	
F	E	G	C	G	G	F	

The columns of this table list the 7 lines, and for each one, the 3 points on the line. Look closely at the table. Any two capital letters occur together in exactly one column (any two points lie on exactly one line) and any two columns have exactly one capital letter in common (any two lines intersect in exactly one point).

There is no longer any need to go back to the original vectors. Every geometric question for $\wp(\mathbb{Z}_2)$ can be answered by just examining this table.

Question 1: Does B lie on the line g?

Answer: No, since B is not in column g.

Question 2: What is $c \cap f$?

Answer: G, since only G is in both columns.

Question 3: What is the line AD?

Answer: b since that is the column where both A and D occur.

Question 4: Are the points A, F, G collinear?

Answer: Yes, since they all lie on the line f.

Question 5: Are the lines b, c, g concurrent?

Answer: No, since there is no letter common to all three columns.

Question 6: Are the triangles ΔCDE and ΔAFG in perspective from a point?

Answer: Yes, they are in perspective from B since $\{B, D, F\}$, $\{B, C, A\}$ and $\{B, E, G\}$ are three sets of collinear points.

Question 7: Are the triangles $\triangle CDE$ and $\triangle AFG$ in perspective from a line?

Answer: Yes, by question 6 and Desargues' Theorem.

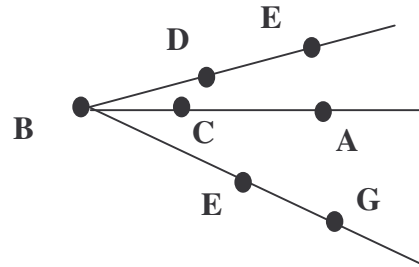
Question 8: Which line?

Answer: $CD \cap AF = c \cap f = G$

$CE \cap AG = g \cap f = F$

$DE \cap FG = b \cap f = A.$

Note that A, F, G are collinear, as predicted by Desargues' Theorem. From the table we see that this line is f . So the triangles are in perspective from the line f .



In the above triangle no attempt has been made to put the points “in the right places” – “right places” doesn’t make sense for the 7-point projective plane. In fact we have had to place E in two places. The diagram is merely an aid for us to remember the three collinear sets of projective points.

Note that there is a certain amount of degeneracy in this example in that the triangle A, F, G consists of 3 collinear points. It is impossible in this tiny example with only 7 points to avoid getting a degenerate triangle. Nevertheless Desargues' Theorem still works even if one of the triangles is degenerate.

Question 9: Which of the 7 lines is the ideal line?

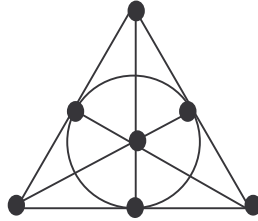
Answer: This is a non-question. Remember that any line can be considered to be the ideal line by placing the relevant affine plane appropriately. For example if we want d to be the ideal line then its three points A, B, C will become the ideal points. Removing the line d and the three points A, B, C we are left with the 4 “ordinary” points D, E, F, G and the 6 “ordinary” lines a, b, c, e, f, g . There are 2 ordinary points on each ordinary line, as follows:

	a	b	c	e	f	g
D	D	D	D	E	F	E
F	F	E	G	G	G	F

Now we have lines which don’t intersect, and which are therefore considered to be parallel. For example $a \parallel e$. These lines don’t intersect in an ordinary point, but, going back to the original table, they intersect in the ideal point B .

It must be emphasized that this distinction between ideal and ordinary points depended on our arbitrary choice of d as ideal line. Any line could have been chosen and whether a point is ordinary or ideal would change. In the projective plane itself there is no such distinction.

It would be nice if we could draw a picture of the 7-point projective plane. Of course we can't, not in the true sense, because our pictures must live in a Euclidean plane. But the following picture can be useful.



Here we have a diagram with 7 points and 7 lines. OK so one of the lines looks more like a circle, but that's the best we can do in a Euclidean plane. There are 3 points on each line and 3 lines through each point. Each pair of distinct points lies on exactly one line and each pair of distinct lines intersect in exactly one point.

We can consider the projective plane $\mathcal{P}(\mathbf{Z}_3)$ over the field of integers modulo 3. This field has the following addition and multiplication tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

There will now be $3^3 = 27$ vectors. Removing the zero vector we are left with 26 non-zero vectors. But because there are now 2 non-zero scalars each projective point will be spanned by 2 different vectors. So the 26 non-zero vectors in fact only give us $26/2 = 13$ distinct points. And by duality there must be 13 lines. Each line is a 2-dimensional subspace containing $3^2 = 9$ vectors, of which 8 are non-zero. But these will only give us $8/2 = 4$ points on the line because there are 2 non-zero scalars and $\langle 2\mathbf{v} \rangle = \langle \mathbf{v} \rangle$.

So the 13-point projective plane has 13 points and 13 lines with 4 points on each line and 4 lines through each point.

By working through the process of constructing finite projective planes one can obtain, by suitably coding the 1-dimensional subspaces, the following collinearity table. Just for fun we'll use the names of the 13 cards in a suit in a pack of cards to denote the 13 points and the same labels for the 13 lines. The entries at the top of the columns denote the lines and the entries in the body of the table give the 4 points on that line.

A	2	3	4	5	6	7	8	9	10	J	Q	K
9	7	Q	Q	J	10	K	Q	J	A	K	5	4
J	6	8	10	8	A	8	9	10	Q	10	K	A
7	5	5	7	6	8	7	6	5	J	6	9	2
4	A	4	3	3	9	2	2	2	K	4	3	3

Note that every pair of "points" occurs in exactly one column and that every pair of "lines" has exactly one common point.

