1. THE REAL PROJECTIVE PLANE

§1.1. Review of Relevant Linear Algebra

A **vector space** over a field F is a set (with the elements being called **vectors**) together with operations of addition and scalar multiplication (a **scalar** being an element of F) such that certain axioms are satisfied. Vectors are usually printed in bold type, such as **v**, to distinguish them from scalars (but there are times when this distinction between scalars and vectors cannot be maintained).

Practically any mathematical object can be an element of a useful vector space. You can have vector spaces of numbers, of matrices, or of functions. But here we need consider only those vectors of the form $(x_1, \ldots, x_n)$, and then only for the case $n = 3$.

So our vectors will all have the form (x, y, z) with the components x, y, z coming from some field. We will mainly use the field **R**, of real numbers, though at one point we will use various finite fields such as $\mathbf{Z}_p$, the field of integers modulo a prime p.

Vector addition and scalar multiplication are defined in the usual way, component by component.

A **linear combination** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a vector of the form

$$\lambda_1\mathbf{v}_1 + \ldots \lambda_n\mathbf{v}_n \text{ for some scalars } \lambda_1, \ldots, \lambda_n.$$

A linear combination is **non-trivial** if at least one of the scalars is non-zero. A set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is **linearly independent** if $\lambda_1\mathbf{v}_1 + \ldots \lambda_n\mathbf{v}_n = 0$ implies that each $\lambda_i = 0$. Otherwise they are **linearly dependent**.

Note that it is the *set* that is linearly dependent or independent, not the vectors themselves. In other words it refers to the relationship between the vectors. A set of vectors is linearly dependent if there exists a non-trivial linear combination of them which is equal to zero. In such a case one can express one of the vectors as a linear combination of the others. If they are linearly independent this is not possible.

Example 1: Are the vectors $(1, 5, -2), (6, -1, 9), (9, -17, 24)$ linearly independent?
Solution: Sometimes students learn to answer such a question by putting the three vectors into a determinant and concluding, that because the determinant is zero, that the vectors must be linearly dependent. That technique is valid in certain cases but it is best to go back to the definition and use the techniques of solving systems of linear equations. Apart from the fact that that the determinant method only works for n vectors in F^n it is seen by many students to be a piece of hocus-pocus – a formula to be invoked, without understanding. The following technique reveals what is happening, and need involve no more work.

Suppose $a(1, 5, -2) + b(6, -1, 9) + c(9, -17, 24) = (0, 0\ 0)$. This is equivalent to the system of equations:

$$\left.\begin{aligned} a + 6b + 9c &= 0 \\ 5a - b - 17c &= 0 \\ -21 + 9b + 24c &= 0 \end{aligned}\right\}$$

which can be represented by an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 6 & 9 & 0 \\ 5 & -1 & -17 & 0 \\ -2 & 9 & 24 & 0 \end{array}\right).$$

As is usual with systems of homogeneous equations we can omit the column of 0's and simply write the system as:

$$\begin{pmatrix} 1 & 6 & 9 \\ 5 & -1 & -17 \\ -2 & 9 & 24 \end{pmatrix}.$$

To solve this system we carry out a sequence of elementary row operations to put the matrix into echelon form. This simpler matrix represents a system of homogeneous linear equations that is equivalent to the original system (equivalent in the sense of having precisely the same solutions).

In this example we obtain:

$$\begin{pmatrix} 1 & 6 & 9 \\ 5 & -1 & -17 \\ -2 & 9 & 24 \end{pmatrix} \xrightarrow[R_2-5R_1,\ R_3+2R_1]{} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -31 & -62 \\ 0 & 21 & 42 \end{pmatrix} \xrightarrow[R_2\div 2]{} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -31 & -62 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[R_2+31R_2,\ R_2\leftrightarrow R_3]{} \begin{pmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly this system has infinitely many solutions, and hence non-trivial ones. This is not so much because of the row of zeros at the bottom as the fact that we have an equivalent system with fewer equations than variables. In solving the above equivalent system:

$$\left.\begin{aligned} a + 6b + 9c &= 0 \\ b + 2c &= 0 \end{aligned}\right\}$$

we may choose c arbitrarily (and hence non-zero) and use the two equations to calculate the corresponding values of a and b.

The conclusion is that the vectors are linearly dependent. It is not necessary to exhibit an explicit relationship, but if called upon to do this we simply choose a convenient non-zero value of c and calculate the corresponding values of a and b.

For example if we take $c = 1$ we get $b = -2$ from the second equation and hence $a = 12 - 9 = 3$ from the first equation. This gives the non-trivial linear relationship: $3(1, 5, -2) - 2(6, -1, 9) + (9, -17, 24) = (0, 0, 0)$, which can be written as:

$$(9, -17, 24) = -3(1, 5, -2) + 2(6, -1, 9).$$

Example 2: Are the vectors $(2, 5, 9), (3, 1, 0), (5, -2, 7)$ linearly independent?

Solution: Writing them as the columns of a matrix (to get the coefficient matrix of the corresponding system of homogeneous linear equations) we have:

$$\begin{pmatrix} 2 & 3 & 5 \\ 5 & 1 & -2 \\ 9 & 0 & 7 \end{pmatrix} \xrightarrow[R_1\div 2]{} \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 5 & 1 & -2 \\ 9 & 0 & 7 \end{pmatrix} \xrightarrow[R_2-5R_1,\ R_3-9R_1]{} \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -\frac{13}{2} & -\frac{29}{2} \\ 0 & -\frac{27}{2} & -\frac{31}{2} \end{pmatrix} \xrightarrow[R_2\div(-13/2)]{} \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & \frac{29}{13} \\ 0 & -\frac{27}{2} & -\frac{31}{2} \end{pmatrix} \xrightarrow[R_3+(27/2)R_2]{} \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & \frac{29}{13} \\ 0 & 0 & -\frac{31}{2}+\frac{27}{2}\cdot\frac{29}{13} \end{pmatrix}.$$

Since $-\frac{31}{2}+\frac{27}{2}\cdot\frac{29}{13} \neq 0$ this system has a unique solution (the zero one).

Note that if we are doing the calculations by hand we can avoid most fractions if we get leading 1's by subtracting suitable multiples of rows from others rather than by division. This greatly simplifies the arithmetic. Re-doing the above example we can obtain:

$$\begin{pmatrix} 2 & 3 & 5 \\ 5 & 1 & -2 \\ 9 & 0 & 7 \end{pmatrix} \xrightarrow[R_2 - 2R_1]{} \begin{pmatrix} 2 & 3 & 5 \\ 1 & -5 & -12 \\ 9 & 0 & 7 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{pmatrix} 1 & -5 & -12 \\ 2 & 3 & 5 \\ 9 & 0 & 7 \end{pmatrix} \xrightarrow[R_2 - 2R_1,\ R_3 - 9R_1]{} \begin{pmatrix} 1 & -5 & -12 \\ 0 & 13 & 29 \\ 0 & 45 & 115 \end{pmatrix} \xrightarrow[R_3 - 3R_2]{} \begin{pmatrix} 1 & -5 & -12 \\ 0 & 13 & 29 \\ 0 & 6 & 28 \end{pmatrix} \xrightarrow[R_2 - 2R_3]{} \begin{pmatrix} 1 & -5 & -12 \\ 0 & 1 & -27 \\ 0 & 6 & 28 \end{pmatrix} \xrightarrow[R_3 - 6R_2]{}$$

$$\begin{pmatrix} 1 & -5 & -12 \\ 0 & 1 & -27 \\ 0 & 0 & 190 \end{pmatrix}.$$

As before this system has only the trivial solution so the vectors are linearly independent.

Example 3: Is the vector $(3, 18, -4)$ a linear combination of the vectors $(1, 5, -3)$ and $(4, 22, 1)$?

Solution: The question is whether there exist scalars x, y such that:

$$(3, 18, -4) = x\,(1, 5, -3) + y\,(4, 22, 1).$$

We can write this equation as a non-homogeneous system of linear equations:

$$\left.\begin{aligned} x + 4y &= 3 \\ 5x + 22y &= 18 \\ -3x + y &= -4 \end{aligned}\right\}$$

Now we can write this system as an augmented matrix: $\left(\begin{array}{cc|c} 1 & 4 & 3 \\ 5 & 22 & 18 \\ -3 & 1 & -4 \end{array}\right)$ and carrying out a sequence of elementary row operations we put this in echelon form:

$$\left(\begin{array}{cc|c} 1 & 4 & 3 \\ 5 & 22 & 18 \\ -3 & 1 & -4 \end{array}\right) \xrightarrow[R_2 - 5R_1,\ R_3 + 3R_1]{} \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 13 & 5 \end{array}\right) \xrightarrow[R_3 - 6R_2]{} \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & -13 \end{array}\right) \xrightarrow[R_2 \leftrightarrow R_3]{} \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 1 & -13 \\ 0 & 2 & 3 \end{array}\right) \xrightarrow[R_3 - 3R_2]{} \left(\begin{array}{cc|c} 1 & 4 & 3 \\ 0 & 1 & -13 \\ 0 & 0 & 29 \end{array}\right).$$

This echelon form represents an equivalent system that is clearly inconsistent. In other words there is no solution for x, y and so $(3, 1, -4)$ is not a linear combination of the vectors $(1, 5, 9)$ and $(4, -2, 1)$.

A set of vectors **spans** a vector space V if every vector is a linear combination of them. A vector space is finite-dimensional if it has a finite spanning set. A set of vectors is a **basis** for V if it is linearly independent and also spans V.

If we have a finite spanning set for V we can remove suitable vectors one by one, as long as the set is linearly independent, without affecting the spanning, until we reach a basis. That is, every spanning set contains a basis. In a similar way, every linearly independent subset of V can be suitably extended to a basis. But any two bases of a finite-dimensional vector space contain the *same number of vectors*. This unique number of vectors in a basis is called the **dimension** of the vector space. The vector space $\mathbf{R}^3$ clearly has dimension 3 since we have the basis $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

If we have a vector space V of dimension n, then any set of more than n vectors must clearly be linearly dependent (otherwise we could extend them to a basis with more vectors than the dimension). And if we have a set of less than n vectors they cannot possibly span V (otherwise we could remove one or more to get a basis of size less than n).

Example 4: Are the vectors (1, 4, 6), (2, 5, 1), (6, 1, 9), (4, 8, 3) linearly independent?
Solution: The answer is clearly NO since you can't have a set of 4 linearly independent vectors inside a 3-dimensional vector space. There is no need to do any calculation in this case other than to count the number of vectors!

A vector space V is said to be the **sum** of subspaces U and W if every vector $\mathbf{v} \in V$ can be expressed as $\mathbf{v} = \mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$. It is said to be the **direct sum** if as well, $U \cap W = 0$. In the first case we write $V = U + W$ and in the case of direct sum we write $V = U \oplus W$. The special feature of a direct sum is that if $V = U \oplus W$ then every vector $\mathbf{v} \in V$ can be expressed *uniquely* as $\mathbf{v} = \mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

If we take a basis for each of the subspaces in a direct sum, and combine them, we will produce a basis for the whole space. This means that $\dim(U \oplus V) = \dim U + \dim V$.

A **Euclidean space** is a vector space V over **R** on which there is defined an **inner-product**, that is a function that assigns to every pair of vectors $\mathbf{u}, \mathbf{v} \in V$ a real number $\mathbf{u}.\mathbf{v}$ such that the following axioms hold:

(1) $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$ for all $\mathbf{u}, \mathbf{v}$;
(2) $\mathbf{u}.(\mathbf{v} + \mathbf{w}) = \mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{w}$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$;
(3) $\mathbf{u}.\mathbf{u} > 0$ if $\mathbf{u} \neq \mathbf{0}$.

Where the space is a space of functions the inner product is often defined as a certain definite integral, but in the simple case of row vectors over **R** we usually take the **dot-product**:

$$(x_1, \ldots, x_n).(y_1, \ldots, y_n) = x_1y_1 + \ldots + x_ny_n.$$

Vectors in a Euclidean space are **orthogonal** if their dot-product is zero. The **orthogonal complement** of a subspace U of a vector space V is defined to be the set of all vectors that are orthogonal to every vector in U, that is $U^\perp = \{\mathbf{v} \in V \mid \mathbf{u}.\mathbf{v} = 0 \text{ for all } \mathbf{u} \in U\}$. The orthogonal complement $U^\perp$ is also a subspace of V and it is easy to show that every vector in V can be expressed uniquely as the sum of a vector in U and a vector orthogonal to every vector in U. In symbols this can be expressed as $V = U \oplus U^\perp$. One can also easily show that for all subspaces of a Euclidean space $U^{\perp\perp} = U$ (the orthogonal complement of the orthogonal complement is the original subspace).

For the purpose of Projective Geometry we will be dealing with $\mathbf{R}^3$ with the usual dot-product. This means that we will only have to consider (apart from the zero subspace and the whole of $\mathbf{R}^3$) subspaces of dimension 1 and 2. Moreover the orthogonal complement of a 1-dimensional subspace will be a 2-dimensional one and vice-versa

Vectors in $\mathbf{R}^3$ have a simple geometric interpretation whereby (x, y, z) denotes the point with coordinates x, y, z in 3-dimensional Euclidean space. A 1-dimensional subspace of $\mathbf{R}^3$ will then represent a line through the origin and a 2-dimensional subspace will represent a plane through the origin.

Often we interpret (x, y, z) as representing not just the point, but the directed line segment from the origin to that point. In this case the vector has both magnitude (length) and direction. With this interpretation, non-zero vectors are orthogonal if and only if they are perpendicular.

The orthogonal complement of a 1-dimensional subspace of $\mathbf{R}^3$ (a line through the origin) is a 2-dimensional subspace (a plane through the origin), being the plane perpendicular to the line. The orthogonal complement of a plane through the origin is a line through the origin.

In this special case of $\mathbf{R}^3$ we have another product of vectors. The **cross product** $\mathbf{u} \times \mathbf{v}$ of two vectors $\mathbf{u}$, $\mathbf{v}$ in a 3-dimensional vector space is a vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. It is defined by:

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) \ = \ (x_2y_3 - x_3y_2, \ -(x_1y_3 - x_3y_1), \ x_1y_2 - x_2y_1).$$

It is more easily remembered as the 3 × 3 "determinant":

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

The top row are the standard basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ and not scalars like the second and third rows. Nevertheless using the usual rules for expanding determinants we obtain a valid expression for the cross-product.

Example 5: If $\mathbf{u} = (1, 5, -3)$ and $\mathbf{v} = (4, -1, 2)$ find $\mathbf{u} \times \mathbf{v}$.

Solution: $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 5 & -3 \\ 4 & -1 & 2 \end{vmatrix} = [5.2 - (-3)(-1)]\,\mathbf{i} - [1.2 - (-3).4)]\,\mathbf{j} + [1.(-1) - 5.4]\,\mathbf{k}$

$= 7\mathbf{i} - 14\mathbf{j} - 21\mathbf{k} = (7, -14, -21).$

Check: $(7, -14, -21).(1, 5, -3) = 7 - 70 + 63 = 0.$

$(7, -14, -21).(4, -1, 2) = 28 + 14 - 42 = 0.$

Since it is possible to make an error in sign at some stage it is important that you check your answer to a cross-product against at least one of the two vectors.

Another concept from Linear Algebra that we will find useful is that of a linear transformation. A function $f : U \rightarrow V$ is a **linear transformation** if

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v).$$

The **kernel** of such a linear transformation is $\ker f = \{u \in U \mid f(u) = 0\}$ and its **image** is $\operatorname{im} f = \{f(u) \mid u \in U\}$. The kernel is a subspace of U and its dimension is called the **nullity** of f. The image is a subspace of V and its dimension is called the **rank** of f. One can show that

$$\operatorname{rank}(f) + \operatorname{nullity}(f) = \dim U.$$

§1.2 The Real Affine Plane

The real Euclidean plane involves a lot of things that can be measured, such as distances, angles and areas. This is referred to as the *metric structure* of the Euclidean plane. But underlying this is the much simpler structure where all we have are points and lines and the relation of a point lying on a line (or equivalently a line passing through a point). This relation is referred to as the *incidence structure* of the Euclidean plane.

The **real affine plane** is simply the Euclidean plane stripped of all but the incidence structure. This eliminates any discussion of circles (these are defined in terms of distances) and trigonometry (these need measurement of angles). It might seem that there is nothing left but this is not the case.

There is, of course, the concept of **collinearity**. Three points are collinear if the unique line joining any two of them passes through the third. Certain theorems of affine geometry state that if there is such and such a configuration and certain triples of points are collinear then a certain other triple is collinear. Another affine concept is that of **concurrence** – three lines passing through a single point.

Then there is the concept of **parallelism**, though we can't define it in terms of lines having a constant distance between them or lines having the same slope.

Definition: Two lines h, k are **parallel** if either they are equal or they do not intersect (meaning that no point lies on them both).

Here are two basic properties of the real affine plane:
(1) Given any two distinct points there is exactly one line passing through both.
(2) Given any two distinct lines there is at most one point lying on both.

There is an obvious similarity between these two properties. But while we can say "exactly one" in property (1) the best we can do in property (2) is "at most one" because there is no common point when the lines are parallel.

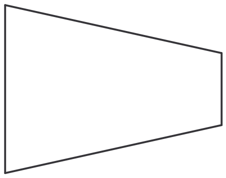
This is reminiscent of the situation with the real number system where every real number (except zero) has at most two square roots. Why not *exactly two*? Because the negative numbers do not have any square roots.

This was considered to be a defect of the real number system and so imaginary numbers were invented to provide square roots for negative numbers. The field of real numbers was expanded to form the complex numbers, and for this system we can say that every non-zero number has *exactly two* square roots.

We do the same thing with the affine plane. Since parallel lines don't meet in the real affine plane we invent "imaginary" points where they do meet – except we call these extra points **ideal points**. The real affine plane is thereby extended to the real projective plane. In the real projective plane there are no longer such things as distinct parallel lines since every pair of distinct lines intersect in exactly one point. This then mirrors exactly the fact that every pair of distinct points lie on exactly one line.

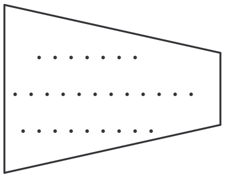
§1.3. Intuitive Construction of the Real Projective Plane

We start with the real affine plane.

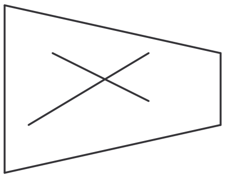


(This trapezium is merely a representative picture of the affine plane. If we drew an accurate picture it would cover the whole page, and then some! There would be no space for any extra points, let alone the text. You should view this as a perspective view of a very large rectangle and then pretend that it extends infinitely far in all directions.)

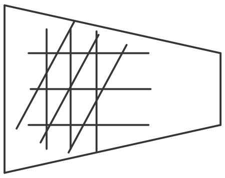
We call the points on the real affine plane **ordinary points**.



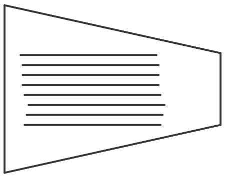
We call the lines on the real affine plane **ordinary lines**.



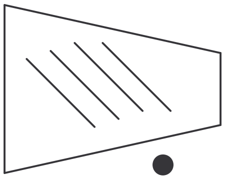
We sort these ordinary lines into parallel classes.



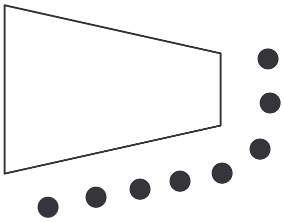
A **parallel class** consists of a line together with all lines parallel to it.



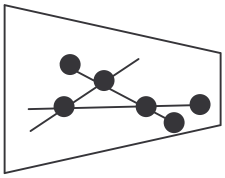
For each parallel class we invent a new point, called an **ideal point**.



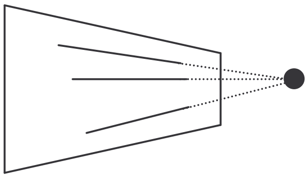
These ideal points do not lie on the real affine plane. Where are they then? The answer is simply "in our minds". However, to assist our imagination, we can put these ideal points on our diagram outside of the shape that represents the real affine plane.



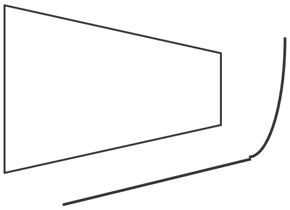
As well as ordinary points lying on ordinary lines in the usual way



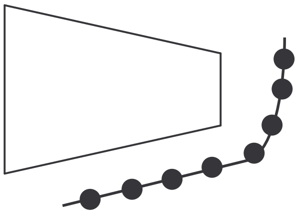
we decree that all lines in a given parallel class (and no others) pass through the corresponding ideal point.



We also invent a new line called the **ideal line**



and decree that this line passes through all the ideal points (and no others).



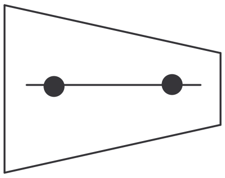
The resulting geometry is called the **real projective plane**. It contains all of the real affine plane, as well as the ideal points and the ideal line. Any theorem that we can prove for the real projective plane will be true for the real affine plane simply by taking the points and lines to be ordinary ones.

§1.4. The Real Projective Plane is Complete

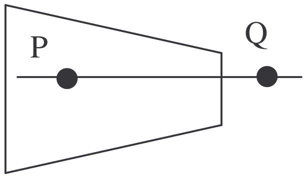
Theorem 1A: In the real projective plane:

(i) any two distinct points lie on exactly on line;
(ii) any two distinct lines intersect in exactly one point.

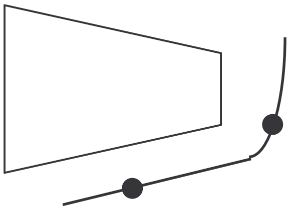
Proof: (i) **Case I:** two ordinary points lie on an ordinary line, as in the affine plane.



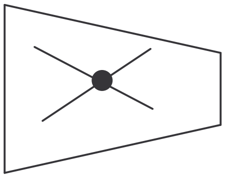
Case II: an ordinary point P and an ideal point Q lie on the line through P parallel to the lines in the parallel class corresponding to Q. (Remember that in Euclidean Geometry there is a unique line which passes through a given point and is parallel to a given line.)



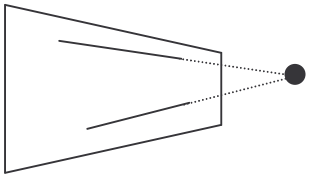
Case III: Two ideal points lie on the ideal line.



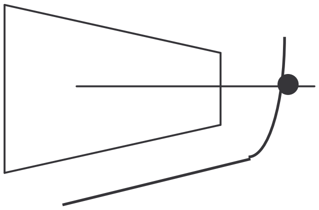
(ii) **Case I: Two ordinary lines:**
If these lines are non-parallel they intersect in an ordinary point.



If they are parallel they intersect in the ideal point that corresponds to their parallel class.



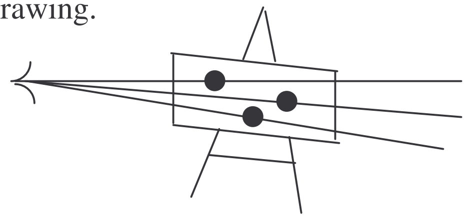
Case II: An ordinary line and the ideal line:
An ordinary line intersects the ideal line in the ideal point corresponding to the parallel class in which it lies.



TRUE/FALSE QUIZ
(1) An ordinary point cannot lie on the ideal line.
(2) An ideal point cannot lie on an ordinary line.
(3) Every ordinary line cuts the ideal line.
(4) Two non-parallel ordinary lines cannot intersect in an ideal point.
(5) There are infinitely many lines parallel to the ideal line.
(6) There are infinitely many parallel classes.
(7) There are infinitely many ideal points.
(8) There are infinitely many ideal lines.
(9) There is exactly one ideal point on every projective line.
(10) There is exactly one ideal line through every projective point.

§1.5. The Artist's View of the Real Projective Plane

Renaissance artists had no problem with the concept of parallel lines meeting a point. This happens all the time in a perspective drawing.



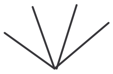
Consider what an artist does when he sketches a scene. You might think that he represents points in the scene by points on the canvas, but it would be more accurate to say that he represents rays not points. Every ray emanating from his eye corresponds to a single point on his canvas.

This leads to the next way of thinking about the real projective plane.

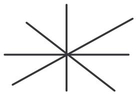
Consider the 3-dimensional Euclidean space, $\mathbf{R}^3$.

Definition: A **projective point** is a line through the origin. A **projective line** is a plane through the origin.

It might seem strange at first to call a line a point and to call a plane a line but to the artist a line in 3-dimensional space, passing through his viewpoint, is what he depicts as a point on his canvas. [This is not completely true, in that an artist (unless he has eyes in the back of his head) only depicts rays (half-lines) not whole lines.]



rays



lines

The real projective plane is defined as the set of all projective points and all projective lines. It can be thought of as a sort of porcupine in 3-dimensional space, bristling with lines going in all directions through the origin. To complete the description we must define incidence, that is, we must explain what it means for a projective point to lie on a projective line.

Definitions:
A projective point P (line through the origin) **lies on** a projective line h (plane through the origin) if, when considered as a line, it lies on h, considered as a plane.
A projective line h **passes through** a projective point P if P lies on h.
Three (or more) projective points are **collinear** if they lie on a common projective line.
Three (or more) projective lines are **concurrent** if they pass through the same projective point.

Theorem 1B: In the real projective plane,

(i) any two distinct projective points lie on exactly on projective line;
(ii) any two distinct projective lines intersect in exactly one projective point.

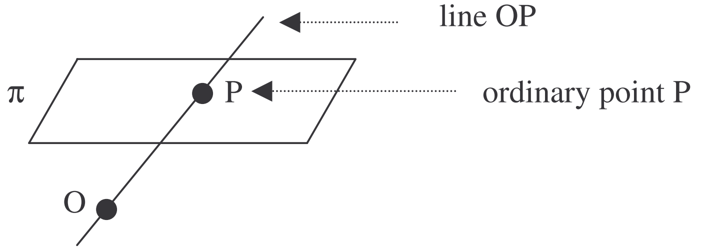
Proof: Translated into ordinary terms these state that:
(i) any two lines through the origin lie on exactly one plane through the origin;
(ii) any two planes through the origin intersect in exactly one line.
Both of these are clearly true statements for 3-dimensional Euclidean Space.

§1.6. Embedding the Real Affine Plane in the Real Projective Plane

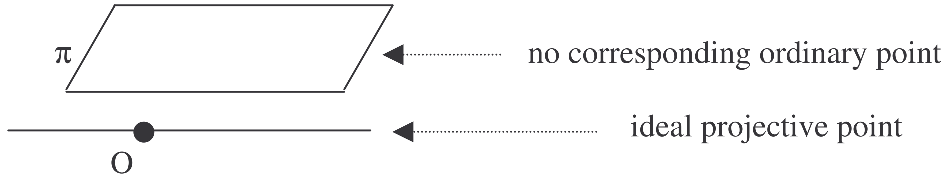
It is all very well to have constructed a geometry whose incidence structure is complete, but is it really the same structure that we created earlier by extending the real affine plane by adding ideal points? After all, with this new version, which projective points are the ideal points? With this bunch of lines through the origin they all seem pretty much the same.

Well, suppose we take an ordinary real affine plane π in $\mathbf{R}^3$, one that doesn't pass through the origin. We can think of the origin as the artist's eye and the plane as his (infinite) canvas.

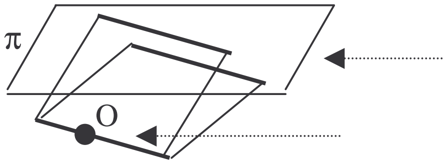
Every (ordinary) point P on π corresponds to a unique line through the origin, namely the line OP. But this is a projective point according to our second point of view.



But if we take the totality of affine points the corresponding projective points (lines through O) do not use up all the available lines through the origin. Only a projective point that, viewed as a line through the origin, cuts π will correspond to a point on π (viz. the intersection of that line with π). Left over will be the projective points (lines through O) that are parallel to π. These are in fact the ideal points in the real projective plane.



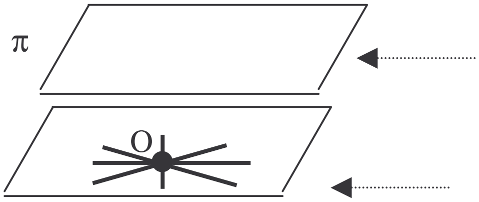
Suppose we have two ordinary parallel lines on π. They do not intersect on π. But the corresponding projective lines *do* intersect. These corresponding projective lines will be the planes through the origin that pass through the respective lines on π. These lines will intersect in a line (just as two pages of a book intersect in the spine of the book).



two parallel ordinary lines

corresponding projective lines intersect in a projective point which does not correspond to an ordinary point

This line will pass through O so can be considered as a projective point. But this line will be parallel to π and so will not correspond to an ordinary point. So two parallel, non-intersecting lines in the ordinary plane π correspond to two projective lines (planes through O) that *do* intersect – in an ideal point (line through O). The ideal points are in fact the lines through O that are parallel to π and the ideal line is thus the plane through O that is parallel to π.



ordinary plane embedded in $\mathbf{R}^3$

the ideal line (corresponding to π)

In this way we have in effect completed the real affine plane to produce the real projective plane. But notice that the plane that becomes the ideal line depends on which affine plane we take in $\mathbf{R}^3$. In fact any projective line can become the ideal line if we take a suitable affine plane π and a suitable origin O.

This is important. The distinction between ordinary point and ideal point is not one that is intrinsic to the real projective plane itself. It only reflects the way we view the real projective plane from the point of view of a real affine plane.

The beauty of the real projective plane is that it is wonderfully uniform. We can prove Euclidean theorems by projective means without the messiness of having to consider cases where lines are parallel and where lines are not parallel. In the real projective plane, all points are equivalent and all lines are equivalent.

Example 6: Suppose we take the plane π to be $z = 1$.
(i) Which of the axes (x-, y- and z-) are ideal points and which are ordinary (relative to π)?
(ii) What is the projective point corresponding to the ordinary point $(1, 1, 1)$ on π?
(iii) What point on π corresponds to the projective point $x = 2y = 3z$?
(iv) Relative to which planes is $x + y + z = 0$ the ideal line?
Solution: (i) The x- and y-axes are ideal, the z-axis is ordinary.
(ii) The line $x = y = z$.
(iii) $(3, 3/2, 1)$ (this is where the line $x = 2y = 3z$ intersects π).
(iv) any plane of the form $x + y + z = c$ for $c \neq 0$.