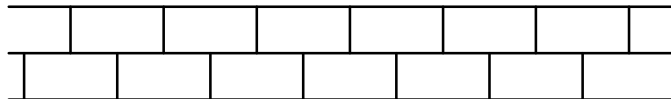


INTRODUCTION TO GROUPS

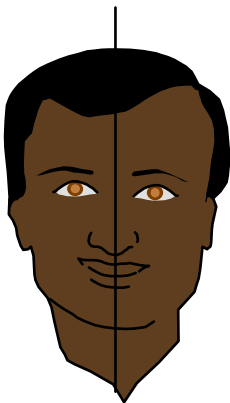
1. SYMMETRY

Symmetry can be found in many places, in art and graphic design, music, architecture, in the natural world and in science. And there are many types of symmetry. There is the mirror symmetry we expect to find in the human face, there is the rotational symmetry such as you find in a honeycomb and the translational symmetry that is found in a repetitive piece of music or a recurring decimal. Sometimes the symmetry is a combination of translational and mirror symmetry as in a pattern of bricks:



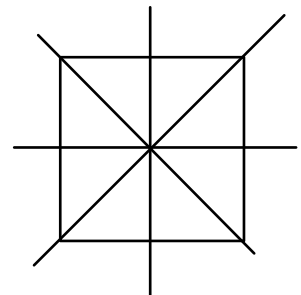
Poetry exhibits aspects of symmetry in its rhyming patterns and physical laws involve symmetry. Even asymmetry makes use of symmetry for its effect in that it relies on our unsatisfied expectation of symmetry for its effect.

But what really is symmetry? The most useful definition is in terms of operations which keep something the same. The human face is never exactly symmetrical, but we imagine it to have mirror symmetry about a vertical axis of symmetry. If reflected left-to-right in this axis a face appears to be the same. The reflection operation is therefore a symmetry operation for the face.



So whenever we have an axis of symmetry we have a symmetry operation. A square has four axes of symmetry and four associated symmetry operations. But a square has another type of symmetry. Rotating a square through 90° , it looks the same. So a 90° rotation about the centre is a symmetry operation.

And if a 90° rotation is a symmetry operation then so are two 90° rotations, that is a 180° rotation, and three lots of 90° , or a 270° rotation. The centre of the square is called a centre of symmetry and is associated with these three rotational symmetry operations.



One feature of symmetry operations that is illustrated by the above example is the fact that if you combine two symmetry operations, one following the other, you always get a symmetry operation.

This leads to a system of symmetry operations. We call the result of combining two symmetry operations their *product*. And denoting operations by letters, just as in ordinary algebra we represent numbers by letters, we write **RS** for the product of **R** and **S**.

If **R** is a 90° anti-clockwise rotation, one of the symmetry operations of the square, then **R**² (by which we mean **RR**) is a 180° rotation. And **R**³ is a 270° anti-clockwise rotation, or a 90° clockwise rotation.

But what is **R**⁴? Well of course it's a 360° rotation, but in terms of the overall effect it's equivalent to a 0° rotation. If somebody rotates an object through 360° and we only get to

see the object before and after the rotation, there is no way we can know that it has been rotated at all.

This is a special symmetry operation. It's the operation of doing nothing at all to the object (or doing something which achieves the same effect in terms of final position). We call it the *identity operation* and it's always denoted by the letter **I** because a capital **I** looks like the number 1 and the identity operation behaves like the number 1. If you multiply any number by 1 you leave it unchanged. If you multiply an operation by the identity operation you likewise leave it unchanged. So $\mathbf{IX} = \mathbf{X} = \mathbf{XI}$ are valid equations for any operation **X**.

Getting back to the 90° rotation, which we called **R**, we note that $\mathbf{R}^4 = \mathbf{I}$. In words this says that the operation **R**, carried out four times in succession, produces the same effect on a square as the identity operation **I**.

2. SYMMETRY GROUPS

The set of all symmetry operations of an object, is called its *symmetry group*. Let's work out a few examples.

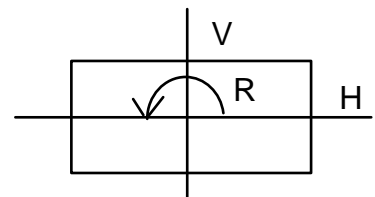
The Letter A:



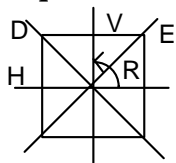
The Symmetry Group of the letter A consists of the identity operation **I** (in fact all symmetry groups will contain **I**) together with the reflection **R** in the vertical axis. If we denote the symmetry group by **G** we have $\mathbf{G} = \{\mathbf{I}, \mathbf{R}\}$.

Rectangle:

The symmetry group of a rectangle consists of **V** (the vertical reflection), **H** (the horizontal reflection), **R** (a 180° rotation) and **I** (the identity). So as a set, the symmetry group **G** is $\{\mathbf{I}, \mathbf{H}, \mathbf{R}, \mathbf{V}\}$.



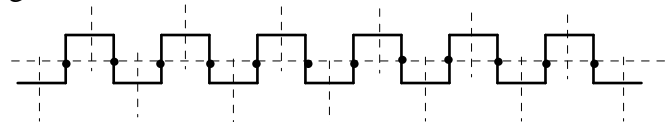
Square:



The symmetry group of a square is $\{\mathbf{I}, \mathbf{R}, \mathbf{R}^2, \mathbf{R}^3, \mathbf{H}, \mathbf{V}, \mathbf{D}, \mathbf{E}\}$ where **H**, **V**, **D**, **E** are the reflections in the axes indicated on the diagram and where **R** is a 90° (anticlockwise) rotation.

Castle Turrets:

The machicolations on a castle wall form the jagged outlines from which archers can fire their arrows. An infinitely long pattern of this type has *translational* symmetry in that if you translate the pattern through a certain distance it remains unchanged — each turret just gets moved on to the next.



There is also *reflectional* symmetry in the infinitely many vertical axes of symmetry (the horizontal axis is not an axis of symmetry). Then there is 180° *rotational* symmetry about the centres indicated by dots. Finally there is what is called *glide* symmetry along the horizontal axis. A glide is a reflection followed by a translation. Reflecting the pattern of turrets in the horizontal axis and then translating half a turret distance, every point on the pattern is moved to an equivalent one. The sine curve also

exhibits this same type of symmetry. The symmetry group of this pattern can be described precisely, but as it's an infinite group we won't bother.

Railway lines:

A set of railway tracks is another infinite repeating pattern. But unlike the sine curve or the castle turrets, the horizontal axis is an axis of reflectional symmetry and not just an axis of glide symmetry.



As well, there are infinitely many vertical axes of symmetry and infinitely many centres of 2-fold, that is 180°, rotational symmetry. And finally there are glides built up from these reflections and translations.

3. MULTIPLICATION TABLES OF SYMMETRY GROUPS

The size of a symmetry group of a shape is a crude measure of its symmetry. A rectangle has a bigger symmetry group than the letter **A** so in that sense it is more symmetric. A square has an even bigger symmetry group and so it is even more symmetric.

But much more information is conveyed about the symmetry by analysing the *structure* of the symmetry group than just counting the number of operations. We can summarise the many relationships between the symmetry operations by compiling its *group table*.

The group table for the symmetry group of the letter **A** is {**I**, **R**} where **I** is the identity operation and **R** is a 180° rotation about the vertical axis. Its multiplication table is:

	I	R
I	I	R
R	R	I

This reflects the fact that the identity operation has no effect when combined with other operations, and two 180° rotations returns the letter **A** to its original position.

The multiplication table for the symmetry group of the rectangle is as follows:

	I	H	R	V
I	I	H	R	V
H	H	I	V	R
R	R	V	I	H
V	V	R	H	I

You read off the information from the multiplication table just as you do with multiplication tables of numbers. To multiply **R** by **H** you look in the row marked **R** and the column marked **H**. There you find **V**. This tells us that **RH = V**, that is a 180° rotation followed by a horizontal reflection is equivalent to a vertical reflection.

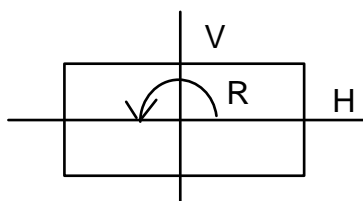
You can check every entry in this table by taking a rectangle (any book will do) and choosing an initial position. Carry out a pair of operations, one after the other, and make a note of the final position. Now return the rectangle to its initial position and carry out the single operation that the table claims to be their product. If the final positions are the same in each case, the equation is correct. (Since it is impractical to reflect the rectangle in an axis of symmetry, you can achieve the same result by flipping it over.)

4. APPLICATION TO MATTRESS TURNING

Next time you are lying in bed, wondering whether to get up, why not reflect on the science of mattress turning. To spread the wear evenly mattresses should be rotated every month. Now the easiest rotation to carry out is the one where you turn the mattress over, left to right. But if you did this every time, the head and feet would never be swapped. So it's necessary to sometimes rotate the mattress in a horizontal plane through 180° .

But thorough mattress turners generally insist that in other months the mattress should be rotated through its third axis. This involves lifting up the head-end up so that the mattress becomes vertical, often narrowly missing the light fitting, and then bringing it down so that it is now at the foot end of the bed. The mattress is now upside down relative to the way it was with head and foot reversed.

Three different operations are possible and among mattress turners (at least the ones who decide which way to rotate each time — not the person at the other end who just does what he's told) the belief is very strong that one should systematically use all three types of rotation, making a note of which way it was last rotated lest the same operation be used twice in a row.



This regime involves a three-monthly cycle such as: **H,R,V,H,R,V...** where **H** is the left-right turn, **R** is the rotation in a horizontal plane and **V** is the very difficult one that nearly breaks the light fitting.

Now there are three things wrong with such a system.

- (1) It's not necessary to do a **V**. From the table we see that **HR = V**, so you can achieve the same result as a **V** without breaking the ceiling light, or your back, simply by doing an **H** and then an **R**.
- (2) You have to remember not only what you did last time but also the time before, and that's not always easy to do.
- (3) It doesn't achieve what it sets out to do (ensuring that the mattress wears evenly) because at the end of each three-month period the mattress is back the way it was. Since there are four possible positions, one gets missed out completely and it's the same one each time! One side of the mattress will get two months wear to every one month on the other side.

So what should all good mattress turners do? Simply leave out the hardest rotation, **V**, and adopt the pattern:

H,R,H,R, ...

But won't this mean that only two of the four positions will get used? Not at all. After two months we have achieved **HR**, that is **V**. After three months the accumulated effect is **HRH = VH = R**. After four months the mattress will be back to its original position, having passed through all four positions! Try explaining that to your mother!

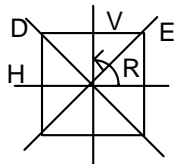
5. THE DIHEDRAL GROUP OF ORDER 8.

The symmetry group of a square is more complicated than that of a rectangle. Cut out a square, number the corners 1 to 4 as in the diagram (number them also 1 to 4 on the back, making sure that each corner has the same number on each side).

Use the following position as the starting position.



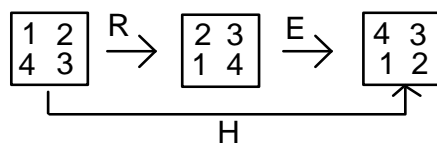
A square has four axes of symmetry, vertical, horizontal and both diagonals. All four axes pass through the centre of the square.



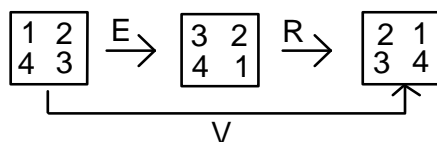
These are axes of 180° rotations belonging to the symmetry group. Let's call them **H**, **V**, **D** and **E**. Now **H** and **V** are the rotations about the horizontal and vertical axes respectively and **D** and **E** are the rotations about the diagonals.

As well as these rotations (which involve flipping the square over) there are rotations in the plane of the square, about the centre. If **R** is a 90° positive (anti-clockwise) rotation about the centre, **R**² is a 180° rotation and **R**³ is a 270° anti-clockwise rotation (or equivalently a clockwise 90° rotation).

The group of symmetries **G** thus consists of eight operations or, as we say, it has *order* 8. The group is $\mathbf{G} = \{\mathbf{I}, \mathbf{R}, \mathbf{R}^2, \mathbf{R}^3, \mathbf{H}, \mathbf{V}, \mathbf{D}, \mathbf{E}\}$. We can use our numbered square as a sort of calculator to perform arithmetic in **G**. For example, $\mathbf{RE} = \mathbf{H}$.



Now let us calculate **ER**. Perhaps you expect the same answer, but watch:

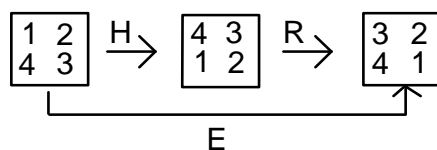


ER is **V**. So $\mathbf{ER} \neq \mathbf{RE}$. The commutative law breaks down in this group.

There are many relationships between these eight elements. We can express them in terms of just two *generators*: **R** and **H**. This means that we can express the others as algebraic expressions built up from just these two.

We've already done this with **R**² and **R**³. It helps to use these names for them rather than pick completely different letters because it reminds us of how they are related to each other and helps with our calculations.

HR is the operation of rotating through 90° and then flipping about the horizontal axis.



As you can see, the net effect is identical to what we would have got if we had simply flipped the square over through 180° about the diagonal axis **E**. We therefore conclude that $\mathbf{E} = \mathbf{HR}$ and so we no longer call it **E** but write it as **HR**.

Similarly $\mathbf{HR}^2 = \mathbf{V}$ and $\mathbf{HR}^3 = \mathbf{D}$. So $\mathbf{G} = \{\mathbf{I}, \mathbf{R}, \mathbf{R}^2, \mathbf{R}^3, \mathbf{H}, \mathbf{HR}, \mathbf{HR}^2, \mathbf{HR}^3\}$. There is nothing special, by the way, about **H**. We could have generated **G** with **R** and **V**, or **R** and **D**, or **R** and **E**. It would even have been possible to generate it with **H** and **D**, or **H** and **E**, though the algebraic expressions would have been more complicated. We could *not* generate all of **G** with just **H** and **V** because these generate the symmetry group of the rectangle, just four of our present eight elements.

We can compute the multiplication table for this group with the help of our little numbered square. But generating the elements in terms of \mathbf{R} and \mathbf{H} gives us another way of doing the arithmetic.

Multiplying \mathbf{R}^2 by \mathbf{R}^3 is easy because it's just \mathbf{R}^5 , and since $\mathbf{R}^4 = \mathbf{I}$, that's just \mathbf{R} . And a calculation such as \mathbf{HR}^2 times \mathbf{R} is easy too. It's just \mathbf{HR}^3 . But what about something like \mathbf{R} times \mathbf{HR}^2 ? Well, of course that's just \mathbf{RHR}^2 . The trouble is that's another name for one of the above eight elements, and we need to identify which one.

Here's where we need to use an important relationship between \mathbf{H} and \mathbf{R} — one that is true for any dihedral group. Suppose we compute \mathbf{RH} and \mathbf{HR}^{-1} , using our numbered square (perhaps for the last time). We get the same result in each case, thus verifying that:

$$\mathbf{RH} = \mathbf{HR}^{-1}$$

This is what we must use instead of the commutative law. We can express this equation in words by saying that:

an H moving past an R, inverts it

So the calculation we started above, but did not finish, is:

$$\mathbf{RHR}^2 = \mathbf{HR}^{-1}\mathbf{R}^2 = \mathbf{HR}.$$

In fact the three relations $\mathbf{R}^4 = \mathbf{I}$, $\mathbf{H}^2 = \mathbf{I}$ and $\mathbf{RH} = \mathbf{HR}^{-1}$ alone are sufficient for us to be able to carry out any calculation in the dihedral group of order 8. We can throw away our numbered square. It's all in those three little relations.

As another example, let us square \mathbf{HR} :

$$(\mathbf{HR})^2 = \mathbf{HR} \times \mathbf{HR} = \mathbf{HRHR} = \mathbf{HHR}^{-1}\mathbf{R} = \mathbf{H}^2 = \mathbf{I}.$$

Because all we need to know about the dihedral group is captured by these three relations, we can express \mathbf{G} very compactly in terms of its generators and relations as:

$$\mathbf{G} = \langle \mathbf{R}, \mathbf{H} \mid \mathbf{R}^4 = \mathbf{H}^2 = \mathbf{I}, \mathbf{RH} = \mathbf{HR}^{-1} \rangle$$

This group reveals something which can never occur with multiplication of numbers: $\mathbf{RH} \neq \mathbf{HR}$. Of course there's no reason why operations should behave like numbers. When you think about it, it usually *does* matter in which order you carry out two operations. For example if " \mathbf{X} = open the door" and " \mathbf{Y} = walk through the door", \mathbf{XY} is usually less painful than \mathbf{YX} . And how many lives have been complicated by the lack of the commutative law when \mathbf{X} = "get married" and \mathbf{Y} = "have a child".

The multiplication table for the dihedral group of order 8 is:

	\mathbf{I}	\mathbf{R}	\mathbf{R}^2	\mathbf{R}^3	\mathbf{H}	\mathbf{HR}	\mathbf{HR}^2	\mathbf{HR}^3
\mathbf{I}	\mathbf{I}	\mathbf{R}	\mathbf{R}^2	\mathbf{R}^3	\mathbf{H}	\mathbf{HR}	\mathbf{HR}^2	\mathbf{HR}^3
\mathbf{R}	\mathbf{R}	\mathbf{R}^2	\mathbf{R}^3	\mathbf{I}	\mathbf{HR}^3	\mathbf{H}	\mathbf{HR}	\mathbf{HR}^2
\mathbf{R}^2	\mathbf{R}^2	\mathbf{R}^3	\mathbf{I}	\mathbf{R}	\mathbf{HR}^2	\mathbf{HR}^3	\mathbf{H}	\mathbf{HR}
\mathbf{R}^3	\mathbf{R}^3	\mathbf{I}	\mathbf{R}	\mathbf{R}^2	\mathbf{HR}	\mathbf{HR}^2	\mathbf{HR}^3	\mathbf{H}
\mathbf{H}	\mathbf{H}	\mathbf{HR}	\mathbf{HR}^2	\mathbf{HR}^3	\mathbf{I}	\mathbf{R}	\mathbf{R}^2	\mathbf{R}^3
\mathbf{HR}	\mathbf{HR}	\mathbf{HR}^2	\mathbf{HR}^3	\mathbf{H}	\mathbf{R}^3	\mathbf{I}	\mathbf{R}	\mathbf{R}^2
\mathbf{HR}^2	\mathbf{HR}^2	\mathbf{HR}^3	\mathbf{H}	\mathbf{HR}	\mathbf{R}^2	\mathbf{R}^3	\mathbf{I}	\mathbf{R}
\mathbf{HR}^3	\mathbf{HR}^3	\mathbf{H}	\mathbf{HR}	\mathbf{HR}^2	\mathbf{R}	\mathbf{R}^2	\mathbf{R}^3	\mathbf{I}

Study this table. You will notice that it is highly structured and follows a definite pattern. For example you will notice that every one of the eight operations appears exactly

once in every row and column. And while the table is superficially symmetric about the top-left-bottom-right diagonal, on closer inspection you will see that it is not. Yet the table does appear to exhibit many other patterns.

5. GALOIS AND HIS GROUPS

The inventor/discoverer of the concept of a group was Evariste Galois a young Frenchman who was fascinated by the inability of mathematicians to discover a formula for solving quintics (polynomial equations involving powers of x up to x^5)— a formula like the quadratic formula which found the solutions in terms of the coefficients using the operations of addition, subtraction, multiplication and division and radicals (the extraction of roots — square roots, cube roots etc).

A formula for the cubic was found in 1515 and the quartic was solved in 1545. The next step was the solution of the general quintic. It wasn't until the early part of the nineteenth century that the Norwegian mathematician, Abel, proved that it couldn't be done. Of course today we have numerical techniques which enable us to solve any equation to any desired degree of accuracy but even now there is no exact formula of the type described. Nor will there ever be because Abel proved that it was a logical impossibility.

Now Galois knew of Abel's work but he wanted to go a stage further. He noted that some quintics can certainly be solved along these lines such as $(x^2 - 5x+1)(x^3 - 3)$. But for others it is not possible. Which ones are soluble by radicals in this way and which are not?

Galois studied algebraic expressions involving the roots $\alpha, \beta, \gamma, \delta, \dots$ of a polynomial such as $E = \alpha\beta + \gamma\delta$. Certain permutations of the roots always leave this value unchanged. For example one could swap α and β or swap γ and δ , or perform both swaps together. Or the pairs α,β and γ,δ themselves could be interchanged. A less obvious permutation would be:

$$\alpha \rightarrow \gamma \rightarrow \beta \rightarrow \delta \rightarrow \alpha$$

which is denoted, in permutation cycle notation, by $(\alpha\gamma\beta\delta)$. This permutation transforms E into $\gamma\delta + \beta\alpha = E$. It leaves E unchanged.

In fact there are in all 8 permutations of $\{\alpha,\beta,\gamma,\delta\}$ which leave the value of E unchanged. These are listed below, together with the corresponding cycle notation.

I:	$\alpha \rightarrow \alpha$	$\beta \rightarrow \beta$	$\gamma \rightarrow \gamma$	$\delta \rightarrow \delta$	I
S:	$\alpha \rightarrow \gamma$	$\beta \rightarrow \delta$	$\gamma \rightarrow \beta$	$\delta \rightarrow \alpha$	$(\alpha\gamma\beta\delta)$
T:	$\alpha \rightarrow \beta$	$\beta \rightarrow \alpha$	$\gamma \rightarrow \delta$	$\delta \rightarrow \gamma$	$(\alpha\beta)(\gamma\delta)$
U:	$\alpha \rightarrow \delta$	$\beta \rightarrow \gamma$	$\gamma \rightarrow \alpha$	$\delta \rightarrow \beta$	$(\alpha\delta\beta\gamma)$
V:	$\alpha \rightarrow \gamma$	$\beta \rightarrow \delta$	$\gamma \rightarrow \alpha$	$\delta \rightarrow \beta$	$(\alpha\gamma)(\beta\delta)$
W:	$\alpha \rightarrow \beta$	$\beta \rightarrow \alpha$	$\gamma \rightarrow \delta$	$\delta \rightarrow \delta$	$(\alpha\beta)$
X:	$\alpha \rightarrow \delta$	$\beta \rightarrow \gamma$	$\gamma \rightarrow \beta$	$\delta \rightarrow \alpha$	$(\alpha\delta)(\beta\gamma)$
Y:	$\alpha \rightarrow \alpha$	$\beta \rightarrow \beta$	$\gamma \rightarrow \delta$	$\delta \rightarrow \gamma$	$(\gamma\delta)$

The multiplication table for this group is:

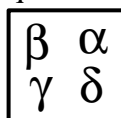
	I	S	T	U	V	W	X	Y
I	I	S	T	U	V	W	X	Y
S	S	T	U	I	Y	V	W	X
T	T	U	I	S	X	Y	V	W
U	U	I	S	T	W	X	Y	V
V	V	W	X	Y	I	S	T	U
W	W	X	Y	V	U	I	S	T
X	X	Y	V	W	T	U	I	S
Y	Y	V	W	X	S	T	U	I

If this group looks vaguely familiar, that is because its essential structure is the same as that of the symmetry group of the square. The only differences are the different names for the elements. If you rename the operations **S** to **Y** as follows:

S → **R** **T** → **R**² **U** → **R**³ **V** → **H** **W** → **HR** **X** → **HR**² **Y** → **HR**³
and translate the multiplication table above it will become a correct table for the symmetry group of the square.

When two groups have the same structure they are said to be *isomorphic*. The square and the expression $E = \alpha\beta + \gamma\delta$ have isomorphic symmetry groups.

The connection between these two examples can be made more transparent by writing the symbols $\alpha, \beta, \gamma, \delta$ on the corners of a square as follows:



The eight operations in the symmetry group of the square are precisely those that preserve the value of $E = \alpha\beta + \gamma\delta$.

Galois associated with every polynomial a group (now called its Galois Group) and described the solubility of polynomials in terms of the structure of this group.

The life of Galois is just as fascinating as his work. In fact it has been the subject of at least one full-length feature film. Galois didn't perform very well at school, got involved in student political riots, did much of his mathematics during his frequent spells in prison, tried unsuccessfully to get the established mathematical community to take notice of his work and was killed in a duel. All by the age of twenty-one!

6. APPLICATIONS OF GROUPS

Group Theory arose to solve the purely mathematical problem of solving polynomial equations. Though this application has tremendous historical importance the need to solve such equations by radical-type formulae has disappeared over the years with the discovery of excellent numerical techniques that enable solutions to be found as accurately as one desires.

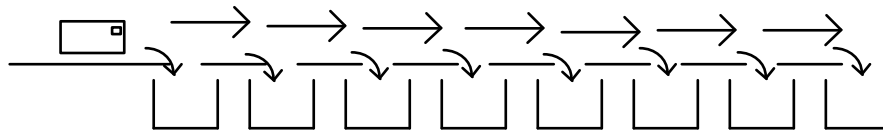
Galois Theory, however, went far beyond the polynomial problem that inspired it and it is used to study mathematical structures called *fields*.

In many other branches of pure mathematics group theory has become an important tool. It is also used widely in quantum physics, in crystallography in chemistry (the study of crystal structures) and in cryptography (encoding confidential data before transmitting it electronically). But since all these applications would require a lot of technical knowledge, as well as a lot more group theory than we can cover here, we shall content ourselves with three rather "low-tech" applications.

(1) Letter Sorting

These days letters are sorted by machine. The postage stamp is, or should be, in the top right-hand corner but many letters will come into the sorting machine upside-down or back-to-front. So the machine must orient the letters all the same way.

Suppose the letters are coming in on a conveyor belt with all possible orientations. We can have a detector which scans the top right-hand corner for a stamp. Those whose stamp is detected are sent off for further processing with the rest being rotated in some way. From here they pass to another detector, and so on. In this way a given letter can be flipped over and rotated until a stamp is detected. And any letter for which no stamp can be found, goes off to another place.

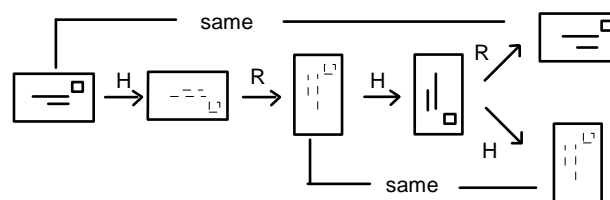


Even though most letters are rectangular rather than square the possible flips and rotations will all be elements of the symmetry group of a square — the dihedral group of order 8.

There are two operations which are widely used: \mathbf{R} = a 90° rotation and \mathbf{H} = a horizontal flip (top to bottom). Although in theory a vertical flip (left to right) could be used, it is only in recent years that vertical flips have been possible at high speed. And flips about a diagonal seem to be quite impractical to implement.

Now it's obvious that a system involving eight detectors and seven flip/rotation operations are necessary. And since it is reasonable to want to minimise the number of operations, we should limit ourselves to just seven operations. But not every sequence of 7 \mathbf{R} 's and \mathbf{H} 's will achieve the desired result of putting a letter through all 8 possible orientations.

Obviously it would be no good having two successive \mathbf{H} 's or four \mathbf{R} 's in a row. And while it might have been patriotic for the British Post Office to start the sequence with \mathbf{HRH} , the next operation would repeat an orientation that has already occurred. This is because \mathbf{HRH} is equivalent to $\mathbf{R}^{-1}\mathbf{H}\mathbf{H} = \mathbf{R}^{-1} = \mathbf{R}^3$ and so if the next operation was \mathbf{R} the letter would repeat its original orientation while if it was \mathbf{H} we would repeat the orientation we had two steps before.



Less obviously, the sequence $\mathbf{RRHRRHR}$ will not do because it repeats two orientations while missing out two others. (Check this yourself.).

A letter-facing sequence in actual use is $\mathbf{RRRHRRR}$. (Check that this achieves all eight orientations). So is $\mathbf{RRHRRRH}$.

Mathematically these are equally good solutions to the problem. But according to Post Office engineers [G.P. Copping "Automatic Letter Facing, British Postal Engineering", *Proceedings of Institution of Mechanical Engineers* (1969-70)] a horizontal flip is easier to implement than a 90° rotation.

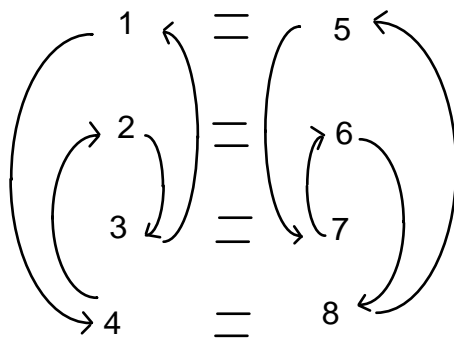
So a sequence such as **RRHRRRH** which involves two **H**'s and only five **R**'s is better than one requiring one **H** and six **R**'s. And if you take into account the fact that an horizontal flip is also quicker than a rotation, a letter-facing sequence with two **H**'s and five **R**'s is better if the **H**'s occur earlier since the average time taken to process a letter would be less.

[An excellent treatment of this application can be found in **J.A. Gallian** "Group Theory and Design of a Letter Facing Machine", **American Mathematical Monthly** vol 84 (1977) 285-287]

(2) Kinship System of the Warlpiri tribe in Northern Australia

It is incredible that such a sophisticated concept as the dihedral group of order 8 should have existed among Australian aborigines for thousands of years. Of course it's misleading to say that the abstract concept existed. But the dihedral group of order 8 is certainly the correct model to explain the complex rules concerning intermarriage in this tribe. Moreover anthropologists have discovered that members of the tribe were able to rapidly perform the necessary calculations required to decide whether or not a given marriage could be allowed — calculations which are equivalent to performing arithmetic in the dihedral group of order 8.

The Warlpiri tribe is divided into eight kinship groups, which for convenience we shall name as 1, 2, ... 8. These eight groups are paired: (1, 5), (2, 6), (3, 7), (4, 8) and the rules involve a diagram such as the following (and they would actually draw such diagrams in the course of explaining their rules to the anthropologists):

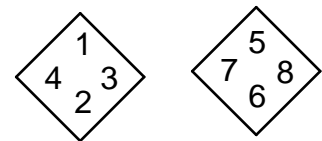


The equal signs show the marriage rules. A man from group 1 could only marry a woman from group 5, and so on. The arrows point from a mother's group to her child's. So any children born to a marriage between a group 1 man and a group 5 woman, was considered to be in group 7. A girl in this family could only marry a group 3 man and her daughters would be in group 6. So a theorem about this system, a fact well-known to the tribe, was that a woman was always in the same group

as her maternal great-great grandmother. And a man was always in the same group as his father's father.

The aborigines didn't have access to cards, but if they did, they would probably have used a square card, numbering the corners on one side as 1,3, 2 and 4 (in order) and the corners on the other side as 5, 7, 6 and 8 as in the following diagram:

Holding the card so that the diagonals run horizontally and vertically, as in a diamond, one could hold the card so that a member of the tribe's group is at the top. Turning it around about the horizontal axis gives the group from which he or she can marry.



And if the wife's group is uppermost, rotating the card clockwise gives the group into which her children are born.

If we denote this mother-child operation by \mathbf{R} , and the marriage operation by \mathbf{H} we get the two generators for the dihedral group of order 8.

The fact that a man's grandson is in his own group is because "son" means "male child of wife", represented by \mathbf{HR} and $(\mathbf{HR})^2 = \mathbf{I}$. The fact that the female line takes four generations to return to the same group is reflected by the fact that $\mathbf{R}^4 = \mathbf{I}$.

[This application is discussed in a recent book by **Marcia Ascher** "Ethno mathematics: A multicultural View of Mathematical View of Mathematics", Belmont, California: Brooks-Cole (1991) reviewed by **Judith Grabiner** in the **American Mathematical Monthly** March 1993.]

(3) Children's Party Game

One of the least useful, but nonetheless amusing, applications of Group Theory is to a children's party game called it "O'Galois Says", in memory of Galois. It's rather a fun sort of game that can be counted on to keep a bunch of bored children amused — for a few minutes anyway. Who said mathematics can't be useful!

O'Galois Says is a game basically like *O'Grady Says* where players are "out" if they make a mistake in obeying the leader's instructions. These instructions refer to the duel in which Galois was killed at the age of 21.

The instructions are **RIGHT**, **LEFT** and **LOAD**. To **LOAD**, you hold your hand up with two fingers outstretched as if holding a pistol. Now here's the catch.

*Whenever the gun is loaded you must do the **opposite** to what you are told.*

If your gun is loaded and you're told to load, you must unload, that is, fire. And if told to turn right with a loaded gun you must turn left and vice versa. But only when the gun is loaded do you do the opposite. At other times you must obey the instructions exactly.

It's quite hilarious to watch when a number of people are playing and you really need to keep your wits about you to play well.

The game is a manifestation of the dihedral group of order 8. The instruction **RIGHT** is equivalent to the generator **R**, the instruction **LEFT** is equivalent to \mathbf{R}^{-1} and the instruction **LOAD** is equivalent to **H**. Just as we got $\mathbf{RH} = \mathbf{HR}^{-1}$ so in duels, $\mathbf{RIGHT} \times \mathbf{LOAD} = \mathbf{LOAD} \times \mathbf{LEFT}$, that is $\mathbf{LOAD} \times \mathbf{RIGHT}^{-1}$.

Of course you don't need to know any group theory to play O'Galois Says. But groups keep popping up in the strangest of places. And just in case you think that the dihedral groups are the only groups worth knowing about, let me emphasise that there are many more fascinating groups than the dihedral ones. The only reason you keep hearing about dihedral groups is the fact that they are reasonably simple to understand without being completely boring!

SUMMARY

Symmetry operation: A *symmetry operation* of an object is a change to the object which leaves it unchanged in some sense.

Symmetry group: The *symmetry group* of an object is the set of all symmetry operations. They are "multiplied" by performing one after the other.

Symmetry group of a shape: The symmetry group of a geometric shape consists of all those movements which leave it in the same overall position. They include *rotations*, *reflections*, *translations* and *glides* (reflections followed by translations).

Symmetry group of an algebraic expression: The symmetry operations of an algebraic expression are permutations of the variables which leave the value of the expression unchanged.

Symmetry group of a square: The symmetry group of a square is the dihedral group of order 8, denoted by D_8 .

Dihedral group of order 8: D_8 can be described using generators and relations as

$$\langle \mathbf{a}, \mathbf{b} \mid \mathbf{a}^4 = \mathbf{b}^2 = \mathbf{1}, \mathbf{b}^{-1}\mathbf{a}\mathbf{b} = \mathbf{a}^{-1} \rangle.$$

In performing arithmetic in the dihedral groups, a \mathbf{b} moving past an \mathbf{a} , inverts it.

Applications: Group Theory began with Galois in connection with the problem of the solubility of the quintic by radicals but now has numerous applications both within mathematics as well as to many outside areas as diverse as chemistry and anthropology.

SKILLS

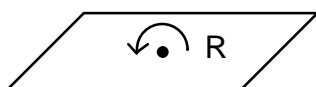
As a result of this chapter you should be able to describe the symmetry operations of geometric shapes and algebraic expressions and, where the number of operations is small, to construct the group table for the symmetry group.

WORKED EXAMPLES

EXAMPLE 1: Find the symmetry group of a parallelogram:

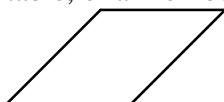


SOLUTION:

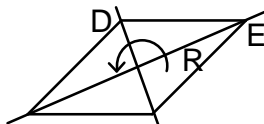


$G = \{I, R\}$ where R is a 180° rotation
 NOTE: A parallelogram has **no** axes of symmetry unless it is a more symmetrical parallelogram such as a rhombus or a rectangle.

EXAMPLE 2: Find the symmetry group, and its group table, of a rhombus:



SOLUTION:



$G = \{I, R, D, E\}$ where R is a 180° rotation about the centre and D, E are 180° rotations about the axes indicated. The group table is:

	I	R	E	D
I	I	R	E	D
R	R	I	D	E
E	E	D	I	R
D	D	E	R	I

EXAMPLE 3: Find the symmetry group of the expression $\alpha\beta - \gamma\delta$.

SOLUTION: The expression is unchanged if one, or both, of the pairs $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are swapped. So the symmetry group is $G =$

$\{I, X, Y, XY\}$ where $X = (\alpha\beta)$, the permutation that swaps a, b and $Y = (\gamma\delta)$.

EXAMPLE 4: Find the symmetry group of the insignia of the Isle of Man:

SOLUTION:

$G = \{I, R, R^2\}$ where R is a 120° rotation about the centre and R^2 is a 240° rotation.

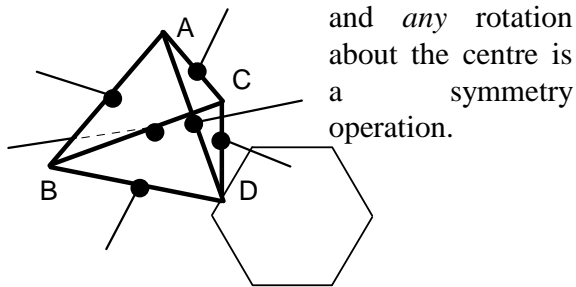
EXAMPLE 5: Classify the letters A–O of the alphabet by their symmetry groups (use the most symmetric possible way of writing each letter).

SOLUTION: A, B, C, D, E, K, L, M each has one axis of symmetry (vertical for A and M, diagonal for L assuming both arms have the same length, and horizontal for the others) so their symmetry groups are $\{I, R\}$ where R is a 180° rotation. The letter N also has symmetry group $\{I, R\}$ but this time R is a 180° rotation about the centre.

The letters G and J have "no symmetry", but since everything has the identity operation as a symmetry operation, their symmetry group is just $\{I\}$.

The letters H and I have the same symmetry as a rectangle: $\{I, H, V \text{ and } R\}$ where H, V and R are a 180° rotations about the horizontal axis, the vertical axis and the centre, respectively.

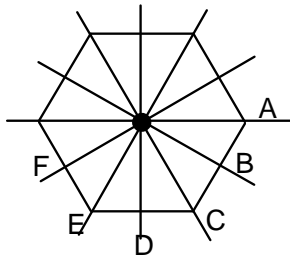
The letter O, represented by a circle, has an infinite symmetry group. Any line through the centre is an axis of symmetry



and *any* rotation about the centre is a symmetry operation.

EXAMPLE 6: Find the symmetry group of a regular hexagon:

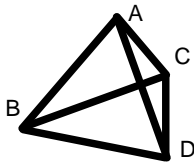
SOLUTION:



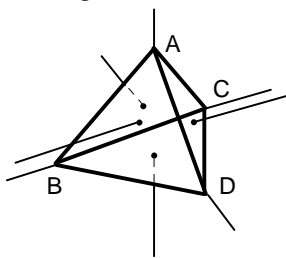
$G = \{I, R, R^2, R^3, R^4, R^5, A, B, C, D, E, F\}$ where R is a 60° rotation about the centre and $A-F$ are 180° rotations about the six axes

of symmetry.

EXAMPLE 7: Find the rotational symmetry operations of a tetrahedron (triangular



pyramid with four identical equilateral triangular faces):



SOLUTION:

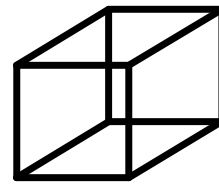
There is 3-fold symmetry (both 120° rotations and 240° rotations about

each of the four axes from a vertex to the midpoint of the opposite face). Less obvious is the 2-fold symmetry (180° about the three axes which join the midpoint of each edge to the midpoint of its opposite edge).

The rotational symmetry group thus has 12 elements:

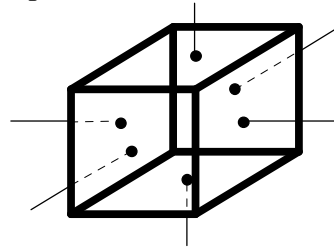
- 1 identity
- 8 3-fold rotations (2 about each of 4 axes)
- 3 2-fold rotations (1 about each of 3 axes)

EXAMPLE 8: Find the symmetry group of a cube:



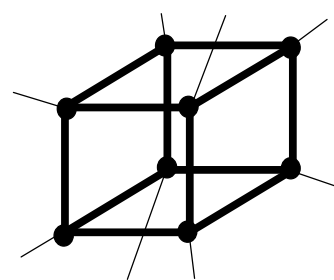
SOLUTION:

The most obvious rotational symmetry operations are the 4-fold rotations about the

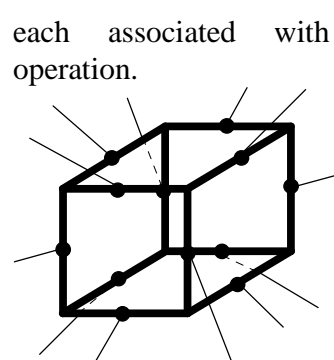


three axes that join the centre of one face to the centre of the opposite face. For each such axis we have

three rotations: 90° , 180° and 270°). This makes 9 symmetry operations in all.



Then there are the 2-fold rotations about the axes that join the midpoints of the sides. There are 6 such axes, each associated with one symmetry operation.



Finally there are the rotations about the three diagonals joining each vertex to the opposite vertex.

If you examine the three edges that come out of each vertex you will see that there is 3-fold rotational symmetry about these diagonal axes. That is, a 120° or a 240° rotation about one of these axes returns the cube to a similar orientation. This gives 2 symmetry operations for each of 4 axes, a total of 8 symmetry operations altogether.

Altogether we have identified

$9 + 6 + 8 = 23$ operations, plus of course the identity giving a total of 24. This is the size of the rotational symmetry group of the cube. There are an additional 24 symmetry operations that arise from reflections.

EXAMPLE 9 Find the symmetries of an infinite row of letter A's:

.....**AAAAAAAAA**.....

SOLUTION: This pattern has translational symmetry. Translate, or move it along through the distance of one letter and the infinite pattern does not appear to have changed. If this operation is denoted by **T**, then the symmetry group also contains **T**², translating it through two A's, **T**³ and so on. Also included is **T**⁻¹, a translation through one letter **A** in the other direction etc. So the group contains **T**ⁿ for any integer n.

As well, there is the 180° symmetry about the vertical axes down through the centre of each **A**, and also the vertical axes separating one **A** from the next.

EXAMPLE 10: Find the symmetry group of

the expression $\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

SOLUTION:

Clearly the permutation **a** = (xyz) which sends $x \rightarrow y, y \rightarrow z, z \rightarrow x$ is a symmetry operation for this expression. So

the symmetry group contains **I, a** and **a**². At first sight the other three permutations on the set {x, y, z}, such as (xy), swapping x and y but fixing z, appears not to be a symmetry operation because it transforms the expression to $\frac{y}{x} + \frac{x}{z} + \frac{z}{y}$ which looks different but in fact is identically equal to the original. In fact the expression can be rewritten as $\frac{xy + yz + zx}{xyz}$ from which we can see that it is fully symmetric in x,y,z. That is, as well as the above three symmetry operations we have (xy), (xz) and (yz), each of which swaps two symbols and fixes the third.

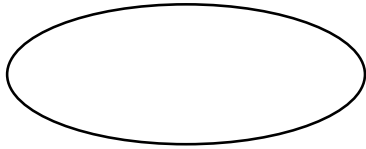
EXAMPLE 11: Find the symmetry group of the following shape:



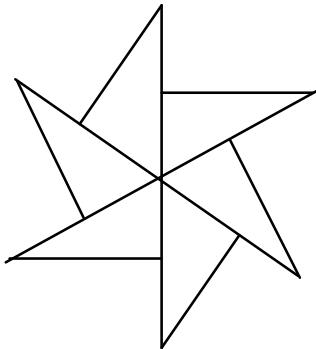
SOLUTION: The symmetry group is {**I, R, R**², **A, B, C**} where **R** is a 120° rotation about the centre, and **A, B, C** are 180° rotations about the three axes of symmetry. This shape has the same symmetry group as the equilateral triangle.

EXERCISES

(1) Find the symmetry group of an ellipse and construct its group table.



(2) Find the symmetry group of the following shape and construct its group table.



(3) Classify the letters P - Z of the alphabet by their symmetry groups (use the most symmetric way of writing each letter).

(4) Find the symmetry group of the expression:

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}$$

and construct its group table.

(5) Describe the symmetry of an infinite row of S's:

(6) Describe the symmetry group of a rectangular box where all three dimensions are different.

(7) Describe the symmetry group of the determinant expression:

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

(8) Describe the symmetries of

(a) the pattern of a normal chessboard;

(b) an infinite chessboard pattern.

(9) Explain the difference between the symmetries of the graphs of $f(x) = \sin x$ and $g(x) = \tan x$.

(10) What are the symmetry groups of the following expressions for vectors in \mathbf{R}^3 .

(a) $\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}$

(b) $\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u}$