

8. LAGRANGE MULTIPLIERS

8.1 Preamble

Given a function $z = f(x, y)$ defined for all x and y , we saw in §5.7 how to locate the stationary points of z (but not to identify their nature). There are many situations in the real world when we may wish to restrict (or constrain) the points (x, y) to those lying on a curve, remembering that a straight line is a special type of curve.

For example, suppose a circular metal wire $x^2 + y^2 = 1$ is heated so that $T(x, y) = x^2 + 2y^2 - x$ is the temperature of the wire at (x, y) . Where are the hottest and coldest points on the wire?

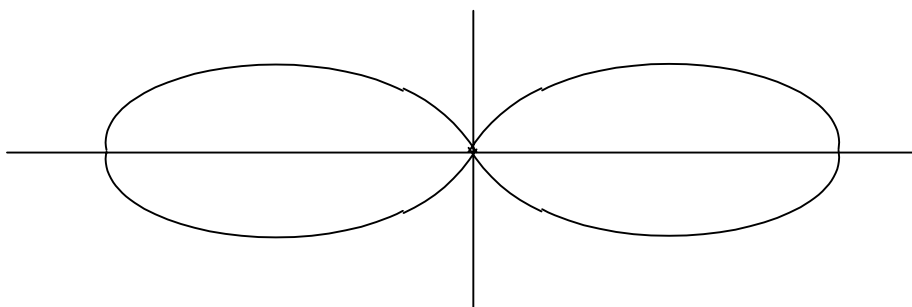
To answer this question, we must find where T has a maximum and minimum value subject to the constraint $x^2 + y^2 = 1$.

[If we ignore the constraint, we see that the only point where $\frac{\partial T}{\partial x} = 0$ and $\frac{\partial T}{\partial y} = 0$ is $(\frac{1}{2}, 0)$, which is not on the circle.]

In this example we can replace y^2 by $1 - x^2$ and T becomes $x^2 + (1 - x^2) - x = 2 - x - x^2$, a function of one variable, with $-1 \leq x \leq 1$. The derivative $\frac{dT}{dx} = -1 - 2x$ so there is a stationary point at $x = -\frac{1}{2}$. Substituting into $x^2 + y^2 = 1$ we get $y = \pm \frac{\sqrt{3}}{2}$. The value of T at these two stationary points is $2\frac{1}{4}$. At the end-points $x = -1$ and $x = 1$ the values of T are 2 and 0 respectively. Therefore the minimum value of T is 0 at $(1, 0)$ and the maximum value is $2\frac{1}{4}$ at $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$.

But suppose that T had instead been $T = x^2 + 2y - x$. The substitution for y would have been much trickier because $y = \sqrt{1 - x^2}$ on the upper half of the circle and $y = -\sqrt{1 - x^2}$ on the lower half. Any attempt to eliminate x , leaving T as a function of y , is equally tricky.

And for more complicated curves, such as a “figure of eight” lying on the x -axis:



whose equation is $(x^2 + y^2)^2 = x^2 - y^2$, it may not be possible to eliminate either x or y to obtain an expression for T that depends only on the other variable.

Note that the equation $y = f(x)$ can be written $y - f(x) = 0$, $x^2 + y^2 = 1$ can be written $x^2 + y^2 - 1 = 0$ and $(x^2 + y^2)^2 = x^2 - y^2$ can be written $(x^2 + y^2)^2 - x^2 + y^2 = 0$. In general a curve in the x - y plane can be represented by an equation of the form $\phi(x, y) = 0$. When we restrict attention to those points on the curve we call its equation a **constraint**.

To overcome the difficulties indicated above of optimising (i.e. maximising or minimising) a function subject to a constraint, there's an elegant device called Lagrange multipliers, named after the French mathematician Joseph Lagrange (1736 – 1813). This topic is not normally taught in elementary courses but it is not difficult and it is used in some economics courses.

8.2 Lagrange Multipliers: Two Variables — One Constraint

Given a function $z = f(x, y)$ and a constraint $\varphi(x, y) = 0$, we create the new function $F(x, y, \lambda) = f(x, y) - \lambda\varphi(x, y)$. This is a function of three variables, the two original variables x and y plus a new one that we call λ (lambda). For the points we are restricted to, those where $\varphi(x, y) = 0$, the values of F are the same as the values of f , irrespective of the value of λ .

We calculate $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$. The solution to our optimisation problem is obtained by solving simultaneously the three equations:

$$\left. \begin{array}{l} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ \varphi(x, y) = 0 \end{array} \right\}.$$

Example 1: Find the maximum and minimum values of $4xy$ subject to the constraint $x^2 + y^2 = 1$.

Solution:

Here $f(x, y) = 4xy$ and $\varphi(x, y) = x^2 + y^2 - 1$.

We define $F(x, y, \lambda) = 4xy - \lambda(x^2 + y^2 - 1)$.

$$\frac{\partial F}{\partial x} = 4y - 2\lambda x \quad \text{and} \quad \frac{\partial F}{\partial y} = 4x - 2\lambda y.$$

So we solve the equations:

$$\left. \begin{array}{l} 4y - 2\lambda x = 0 \\ 4x - 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{array} \right\}$$

From the first two equations we get $y = \frac{1}{2}\lambda x = \frac{1}{2}\lambda(\frac{1}{2}\lambda y) = \frac{1}{4}\lambda^2 y$ so $(\frac{1}{4}\lambda^2 - 1)y = 0$.

So either $y = 0$ or $\frac{1}{4}\lambda^2 = 1$.

If $y = 0$ then $x = 0$, but this does not satisfy the equation $x^2 + y^2 = 1$ and must be rejected.

So $\frac{1}{4}\lambda^2 = 1$, which leads to $\lambda = \pm 2$.

If $\lambda = 2$ then $y = x$. Substituting into the third equation gives $2x^2 = 1$ so $x = \pm \frac{1}{\sqrt{2}}$ and $y = \pm \frac{1}{\sqrt{2}}$.

If $\lambda = -2$ then $y = -x$. Substituting into the third equation again gives $2x^2 = 1$.

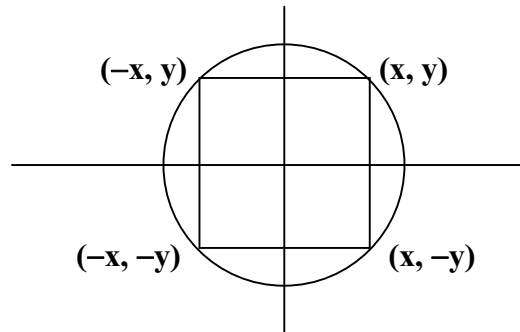
So the maximum values of $4xy$, subject to $x^2 + y^2 = 1$ occur at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. This

maximum value of $4xy$ is 2. Similarly the minimum value is -2 , occurring at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Example 2: What is the area of the largest square that can be inscribed in a circle of radius 1?

Solution: For convenience we take the centre of the circle to be the origin. Its equation is thus: $x^2 + y^2 = 1$. Clearly the largest square will also be centred at the origin.



Let the four corners of the square be $(\pm x, \pm y)$ as shown. The area of the square is $4xy$. So we want to maximise $4xy$ subject to the constraint $x^2 + y^2 = 1$. This we did above and found that the maximum value is 2.

Now it could be argued that common sense would suggest that the problem is symmetric in x and y and that the maximum area would occur when $x = y$. But in the next example this symmetry doesn't exist and so we need the method of Lagrange multipliers.

Example 3: What is the largest square that can be inscribed in the ellipse $x^2 + 2y^2 = 1$?

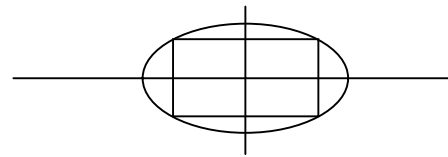
Solution: Here we want to maximise $4xy$ subject to the constraint $x^2 + 2y^2 = 1$.

We define $F(x, y, \lambda) = 4xy - \lambda(x^2 + 2y^2 - 1)$.

$$\frac{\partial F}{\partial x} = 4y - 2\lambda x \text{ and } \frac{\partial F}{\partial y} = 4x - 4\lambda y.$$

So we solve the equations:

$$\left. \begin{aligned} 2y &= \lambda x \\ x &= \lambda y \\ x^2 + 2y^2 &= 1 \end{aligned} \right\}.$$



From the first two equations we get $y = \frac{1}{2} \lambda x = \frac{1}{2} \lambda (\lambda y) = \frac{1}{2} \lambda^2 y$ so $(\frac{1}{2} \lambda^2 - 1)y = 0$.

So either $y = 0$ or $\frac{1}{2} \lambda^2 = 1$.

If $y = 0$ then $x = 0$, but this does not satisfy the equation $x^2 + 2y^2 = 1$ and so must be rejected.

So $\frac{1}{2} \lambda^2 = 1$, which leads to $\lambda = \pm \sqrt{2}$.

If $\lambda = \sqrt{2}$ then $y = \frac{x}{\sqrt{2}}$. Substituting into the third equation gives $\frac{3}{2} x^2 = 1$ so $x = \pm \frac{\sqrt{2}}{\sqrt{3}}$ and

$$y = \pm \frac{1}{\sqrt{3}}.$$

If $\lambda = -\sqrt{2}$ then again we get $\frac{3}{2} x^2 = 1$.

So the maximum values of $4xy$, subject to $x^2 + 2y^2 = 1$ occurs at $\left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

This maximum value is $\frac{4\sqrt{2}}{3}$.

Example 4: Minimise $x^2 + y^2$ subject to the constraint $xy = 1$.

Solution: Let $F(x, y, \lambda) = x^2 + y^2 - \lambda(xy - 1)$.

$$\frac{\partial F}{\partial x} = 2x - \lambda y \quad \text{and} \quad \frac{\partial F}{\partial y} = 2y - \lambda x.$$

So we solve the equations:

$$\left. \begin{aligned} x &= \frac{\lambda}{2} y \\ y &= \frac{\lambda}{2} x \\ xy &= 1 \end{aligned} \right\}.$$

From the first two equations we get $x = \frac{\lambda^2}{4} x$.

Clearly $x \neq 0$ (since $xy = 1$) so $\lambda = \pm 2$.

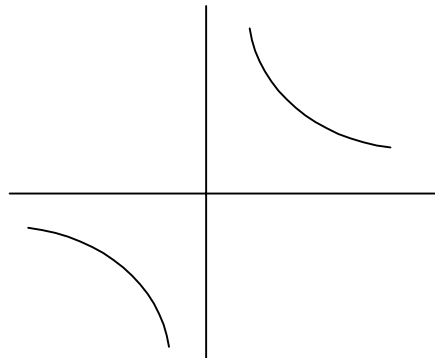
If $\lambda = 2$ then $x = y$. Substituting into the third equation gives $x^2 = 1$ so $x = y = 1$ or $x = y = -1$.

The minimum values of $x^2 + y^2$ is thus 2.

There is no maximum since we can make $x^2 + y^2$ as large as we like by simply taking x sufficiently large, for example when $x = 100$, $y = 0.01$ we have $x^2 + y^2 = 10000.0001$.

Example 5: What is the shortest distance between the curve $xy = 1$ and the origin?

Solution: We want to minimise $\sqrt{x^2 + y^2}$ subject to $xy = 1$. But the point that minimises $\sqrt{x^2 + y^2}$ will also minimise $x^2 + y^2$. So we seek to minimise $x^2 + y^2$ subject to the constraint $xy = 1$.



But we did this above. The minimum value is 2, at $(1, 1)$ and $(-1, -1)$. So the minimum distance is $\sqrt{2}$.

Again it could be objected that this is obvious by symmetry. That's true. We used a very simple example to illustrate the method. But in more complicated cases the answer is not obvious and the Lagrange Multiplier Method is really needed.

Example 6: Maximise $x^2y + 2y$ subject to the constraint $x^2 + y^2 = 1$.

Solution: Let $F(x, y, \lambda) = x^2y + 2y - \lambda(x^2 + y^2 - 1)$.

$$\text{Then } \frac{\partial F}{\partial x} = 2xy - 2\lambda x \quad \text{and} \quad \frac{\partial F}{\partial y} = x^2 + 2 - 2\lambda y.$$

So we solve the equations:

$$\left. \begin{aligned} x(y - \lambda) &= 0 \\ x^2 + 2 &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned} \right\}.$$

From the first equation we get $x = 0$ or $y = \lambda$.

If $x = 0$ then from the third equation $y = \pm 1$. This gives the points $(0, 1)$ and $(0, -1)$. The corresponding values of $x^2y + 2y$ are 2 and -2 .

If $y = \lambda$ then from the second equation $x^2 = 2\lambda^2 - 2$. Substituting into the third equation we get $x^2 + y^2 = 2\lambda^2 - 2 + \lambda^2 = 1$. From this we deduce $3\lambda^2 = 3$ and so $\lambda^2 = 1$. Hence $y = \lambda = \pm 1$. We substitute into $x^2 = 2\lambda^2 - 2$ to get $x = 0$. We have already considered this case. The maximum and minimum are at the points $(0, 1)$ and $(0, -1)$ that we found earlier.

So the maximum value of $x^2y + 2y$, subject to the constraint $x^2 + y^2 = 1$ is 2, occurring at the point $(0, 1)$ and the minimum value is -2 , occurring at $(0, -1)$.

Example 7: A company manufactures things called *widgets* from two raw materials *alandrite* and *beconium*. The number of widgets produced, W , is given by $W = 200a^{1/2}b^{1/4}$ where a and b are the numbers of tonnes of alandrite and beconium respectively. [Such an equation is known by Economists as a *Cobb-Douglas function*.]

Suppose alandrite costs \$50 per tonne and beconium costs \$100 per tonne. Find the maximum number of widgets that can be made if no more than \$150 can be spent on raw materials.

Solution: We wish to maximise $200a^{1/2}b^{1/4}$ subject to the constraint $50a + 100b = 150$.

Let $F(a, b, \lambda) = 200a^{1/2}b^{1/4} - \lambda(50a + 100b - 150)$.

Then $\frac{\partial F}{\partial a} = 100a^{-1/2}b^{1/4} - 50\lambda$, $\frac{\partial F}{\partial b} = 50a^{1/2}b^{-3/4} - 100\lambda$.

We solve the system:

$$\left. \begin{aligned} 100a^{-1/2}b^{1/4} &= 50\lambda \\ 50a^{1/2}b^{-3/4} &= 100\lambda \\ 50a + 100b &= 150 \end{aligned} \right\}.$$

These can be simplified to:

$$\left. \begin{aligned} 2a^{-1/2}b^{1/4} &= \lambda \\ a^{1/2}b^{-3/4} &= 2\lambda \\ a + 2b &= 3 \end{aligned} \right\}.$$

Multiplying the first two equations we can eliminate the variable a :

$$2b^{-1/2} = 2\lambda^2 \text{ so } b^{-1/2} = \lambda^2 \text{ and hence } b = \frac{1}{\lambda^4}.$$

Cubing the first equation gives $8a^{-3/2}b^{3/4} = \lambda^3$. Multiplying this by the second equation gives

$$8a^{-1} = 2\lambda^4. \text{ Hence } a = \frac{4}{\lambda^4}.$$

Now we substitute $a = \frac{4}{\lambda^4}$ and $b = \frac{1}{\lambda^4}$ into the third equation to get:

$$\frac{4}{\lambda^4} + \frac{2}{\lambda^4} = 3.$$

Hence $\frac{6}{\lambda^4} = 3$ and so $\lambda^4 = 2$. Thus $a = \frac{4}{\lambda^4} = 2$ and $b = \frac{1}{\lambda^4} = \frac{1}{2}$.

So the manufacturer should use 2 tonnes of alandrite and half a ton of beconium. The corresponding value of $W = 200a^{1/2}b^{1/4} \approx 237.841423$. So the manufacturer can make about 238 widgets, more than by spending the \$150 on any other combination of the raw materials.

8.3 Lagrange Multipliers: Three or More Variables — One Constraint

If we wish to find the maximum or minimum value of $f(x, y, z)$ subject to the constraint $\varphi(x, y, z) = 0$ we consider $F(x, y, z, \lambda) = f(x, y, z) - \lambda\varphi(x, y, z)$. We then solve the following system of equations simultaneously:

$$\left. \begin{array}{l} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ \frac{\partial F}{\partial z} = 0 \\ \varphi(x, y, z) = 0 \end{array} \right\} .$$

Example 8: The surface $x + 2y - 2z = 3$ is a plane in 3-dimensional space. Find the distance of this plane from the origin.

[This is the shortest distance of any point (x, y, z) on the plane from $(0, 0, 0)$.]

Solution: We want to minimise $\sqrt{x^2 + y^2 + z^2}$ subject to the constraint $x + 2y - 2z = 3$. But minimising $\sqrt{x^2 + y^2 + z^2}$ is equivalent to minimising $x^2 + y^2 + z^2$ so we use that instead to avoid the complications of square roots.

Let $F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + 2y - 2z - 3)$.

$$\text{Then } \frac{\partial F}{\partial x} = 2x - \lambda; \quad \frac{\partial F}{\partial y} = 2y - 2\lambda; \quad \frac{\partial F}{\partial z} = 2z + 2\lambda.$$

So we solve the system:

$$\left. \begin{array}{l} 2x = \lambda \\ 2y = 2\lambda \\ 2z = -2\lambda \\ x + 2y - 2z = 3 \end{array} \right\} .$$

The first three equations give $x = \frac{\lambda}{2}$, $y = \lambda$, $z = -\lambda$. Substituting into the last equation gives:

$$\frac{\lambda}{2} + 2\lambda + 2\lambda = 3 \text{ whose solution is } \lambda = 2/3.$$

Hence $x = 1/3$, $y = 2/3$, $z = -2/3$. The point $(1/3, 2/3, -2/3)$ is therefore the point on the plane which is closest to the origin. The distance of this point from the origin is $\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{1} = 1$. So the distance of this plane from the origin is 1.

The method of Lagrange multipliers extends to any number of variables and any number of constraints. Suppose, for example, we wish to maximise or minimise $F(x, y, z)$ subject to the constraints $\varphi(x, y, z) = 0$ and $\theta(x, y, z) = 0$. The equations $\varphi(x, y, z) = 0$ and $\theta(x, y, z) = 0$ each represent a surface in 3-dimensional space. By imposing both constraints we are considering only those point that lie on both surfaces. This is usually a curve — the curve where the surfaces intersect. So we are seeking to maximise or minimise F along this curve.

We let $F(x, y, z, \lambda, \mu) = f(x, y, z) - \lambda\phi(x, y, z) - \mu\theta(x, y, z)$. Then we solve the system:

$$\left. \begin{aligned} \frac{\partial F}{\partial x} &= 0 \\ \frac{\partial F}{\partial y} &= 0 \\ \frac{\partial F}{\partial z} &= 0 \\ \phi(x, y, z) &= 0 \\ \theta(x, y, z) &= 0 \end{aligned} \right\} .$$

However the task of solving the resulting equations becomes more difficult when we have more than one constraint. We therefore keep to a maximum of 3 variables and one constraint in our examples.

EXERCISES

Exercise 1: Use the method of Lagrange Multipliers to find the maximum and minimum values of xy subject to the constraint $x^2 + y^2 = 18$.

Exercise 2: Use the method of Lagrange Multipliers to find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $2x + 3y = 13$.

Exercise 3: Use the method of Lagrange Multipliers to find the maximum and minimum values of $x + 2y$ for (x, y) on the ellipse $x^2 + 2y^2 = 12$.

Exercise 4: Use the method of Lagrange Multipliers to find the maximum and minimum values of xy subject to the constraint $4x^2 + y^2 = 8$.

Exercise 5: Use the method of Lagrange Multipliers to find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^4 + y^4 = 2$.

Exercise 6: Use the method of Lagrange Multipliers to find the maximum and minimum values of $x^3 - y^3$ subject to the constraint $x^2 + y^2 = 8$.

Exercise 7: Use the method of Lagrange Multipliers to find the maximum and minimum values of $x + 2y - 2z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Exercise 8: A drug company manufactures *oxymaxin* from two ingredients which are known simply as ingredient X and ingredient Y. The number of doses of oxymaxin produced, D , is given by the Cobb-Douglas function $D = 6x^{2/3}y^{1/2}$ where x and y are the numbers of grams of ingredient X and ingredient Y respectively.

Suppose ingredient X costs \$4 per gram and ingredient Y costs \$3 per gram. Find the maximum number of doses that can be made if no more than \$7000 can be spent on raw materials.

SOLUTIONS

Exercise 1: Let $F(x, y, \lambda) = xy - \lambda(x^2 + y^2 - 18)$.

Then $\frac{\partial F}{\partial x} = y - 2\lambda x$ and $\frac{\partial F}{\partial y} = x - 2\lambda y$.

We solve the system:

$$\left. \begin{aligned} y &= 2\lambda x \\ x &= 2\lambda y \\ x^2 + y^2 &= 18 \end{aligned} \right\}.$$

From the first two equations $y = 2\lambda x = (2\lambda)(2\lambda y) = 4\lambda^2 y$.

Hence $y = 0$ or $4\lambda^2 = 1$. If $y = 0$ then $x = 0$, but these values don't satisfy the third equation.

Hence $4\lambda^2 = 1$, so $\lambda = \pm \frac{1}{2}$.

If $\lambda = \frac{1}{2}$ then $x = y$. Substituting into the third equation we get $2x^2 = 18$ so $x = y = \pm 3$.

If $\lambda = -\frac{1}{2}$ then $x = -y$. Substituting into the third equation we get $2x^2 = 18$ so $x = 3$ and $y = -3$ or $x = -3$ and $y = 3$.

The maximum is 9, at $x = 3, y = 3$ or $x = -3, y = -3$.

The minimum is -9, at $x = 3, y = -3$ or $x = -3, y = 3$.

Exercise 2: Let $F(x, y, \lambda) = x^2 + y^2 - \lambda(2x + 3y - 13)$.

Then $\frac{\partial F}{\partial x} = 2x - 2\lambda$ and $\frac{\partial F}{\partial y} = 2y - 3\lambda$.

We solve the system:

$$\left. \begin{aligned} 2x &= 2\lambda \\ 2y &= 3\lambda \\ 2x + 3y &= 13 \end{aligned} \right\}.$$

From the first two equations $x = \lambda$ and $y = (3\lambda/2)$.

Substituting into the third equation we get $2\lambda + (9\lambda/2) = 13$, so $\lambda = 2$. Hence $x = 2$ and $y = 3$.

The minimum is 13, at (2, 3). [Since we are essentially optimising the distance of a point on the line $2x + 3y = 13$ from the origin there is no maximum.]

Exercise 3: Let $F(x, y, \lambda) = x + 2y - \lambda(x^2 + 2y^2 - 12)$.

Then $\frac{\partial F}{\partial x} = 1 - 2\lambda x$ and $\frac{\partial F}{\partial y} = 2 - 4\lambda y$.

We solve the system:

$$\left. \begin{aligned} 2\lambda x &= 1 \\ 4\lambda y &= 2 \\ x^2 + 2y^2 &= 12 \end{aligned} \right\}.$$

From the first two equations $x = \frac{1}{2\lambda}$ and $y = \frac{1}{2\lambda}$. So $x = y$.

Substituting into the third equation we get $3x^2 = 12$ so $x = y = \pm 2$.

The maximum is 6, at $x = 2, y = 2$. The minimum is -6, at $x = -2, y = -2$.

Exercise 4: Let $F(x, y, \lambda) = xy - \lambda(4x^2 + y^2 - 8)$.

Then $\frac{\partial F}{\partial x} = y - 8\lambda x$ and $\frac{\partial F}{\partial y} = x - 2\lambda y$.

We solve the system:

$$\left. \begin{aligned} y &= 8\lambda x \\ x &= 2\lambda y \\ 4x^2 + y^2 &= 8 \end{aligned} \right\}.$$

From the first two equations $y = (8\lambda)x = (8\lambda)(2\lambda)y$.

If $y = 0$ then $x = 0$ which doesn't satisfy the last equation. Hence $16\lambda^2 = 1$ so $\lambda = \pm \frac{1}{4}$.

Case I: $\lambda = \frac{1}{4}$. Then $y = 8\lambda x = 2x$.

Substituting into the third equation we get $4x^2 + 4x^2 = 8$. So $x = \pm 1$.

If $x = 1$ then $y = 2$. If $x = -1$ then $y = -2$.

Case II: $\lambda = -\frac{1}{4}$. Then $y = 8\lambda x = -2x$.

Substituting into the third equation we get $4x^2 + 4x^2 = 8$. So $x = \pm 1$.

If $x = 1$ then $y = -2$. If $x = -1$ then $y = 2$.

The maximum is 2, at $x = 1, y = 2$ and at $x = -1, y = -2$.

The minimum is -2, at $x = -1, y = 2$ and at $x = 1, y = -2$.

Exercise 5: Let $F(x, y, \lambda) = x^2 + y^2 - \lambda(x^4 + y^4 - 2)$.

Then $\frac{\partial F}{\partial x} = 2x - 4\lambda x^3$ and $\frac{\partial F}{\partial y} = 2y - 4\lambda y^3$.

We solve the system:

$$\left. \begin{aligned} 2x &= 4\lambda x^3 \\ 2y &= 4\lambda y^3 \\ x^4 + y^4 &= 2 \end{aligned} \right\}.$$

From the first equation $x = 0$ or $4\lambda x^2 = 2$.

Suppose $x = 0$. Then $y^4 = 2$ so $y = \pm 2^{1/4}$.

Suppose $x \neq 0$. Then $4\lambda x^2 = 2$ which gives $x^2 = \frac{1}{2\lambda}$.

From the second equation $y = 0$ or $4\lambda y^2 = 2$.

Suppose $y = 0$. Then $x^4 = 2$ so $x = \pm 2^{1/4}$.

Suppose $y \neq 0$. Then $4\lambda y^2 = 2$ which gives $y^2 = \frac{1}{2\lambda}$.

So if neither x nor y is zero we must have $x^2 = y^2 = \frac{1}{2\lambda}$.

Substituting into the third equation we get $x^4 + x^4 = 2$ so $x^4 = 1$ and hence $x^2 = 1$ and $x = \pm 1$.

If $x = 1$ then $y = \pm 1$ and similarly if $x = -1$.

The maximum is 2 at the four points $(\pm 1, \pm 1)$.

The minimum is $\sqrt{2}$ at the four points $(0, \pm 2^{1/4})$ and $(\pm 2^{1/4}, 0)$.

Exercise 6: Let $F(x, y, \lambda) = x^3 - y^3 - \lambda(x^2 + y^2 - 8)$.

Then $\frac{\partial F}{\partial x} = 3x^2 - 2\lambda x$, $\frac{\partial F}{\partial y} = -3y^2 - 2\lambda y$.

We solve the system:

$$\left. \begin{aligned} 3x^2 &= 2\lambda x \\ y^2 &= -2\lambda y \\ x^2 + y^2 &= 8 \end{aligned} \right\}.$$

If $x = 0$ then $y = \pm \sqrt{8}$ and $x^3 - y^3 = \pm 16\sqrt{8} = \pm 32\sqrt{2}$.

If $y = 0$ then $x = \pm \sqrt{8}$ and $x^3 - y^3 = \pm 16\sqrt{8} = \pm 32\sqrt{2}$.

Suppose neither x nor y is zero.

Then we may divide the first two equations by x and y respectively to get:

$$3x = 2\lambda \text{ and } 3y = -2\lambda.$$

$$\text{This gives } x = \frac{2\lambda}{3} \text{ and } y = -\frac{2\lambda}{3}.$$

Substituting into the third equation we get $\frac{4\lambda^2}{9} + \frac{4\lambda^2}{9} = 8$. Hence $\frac{8\lambda^2}{9} = 8$ and so $\lambda = \pm 3$.

If $\lambda = 3$ then $x = 2$ and $y = -2$. The corresponding value of $x^3 - y^3$ is 16.

If $\lambda = -3$ then $x = -2$ and $y = 2$. The corresponding value of $x^3 - y^3$ is -16.

Comparing these values with those obtained at the points $(\sqrt{8}, 0)$ and $(0, \sqrt{8})$ we see that the maximum is $32\sqrt{2}$ and the minimum is $-32\sqrt{2}$.

Exercise 7: Let $F(x, y, z, \lambda) = x + 2y - 2z - \lambda(x^2 + y^2 + z^2 - 1)$.

$$\text{Then } \frac{\partial F}{\partial x} = 1 - 2\lambda x, \quad \frac{\partial F}{\partial y} = 2 - 2\lambda y \text{ and } \frac{\partial F}{\partial z} = -2 - 2\lambda z.$$

We solve the system:

$$\left. \begin{array}{l} 2\lambda x = 1 \\ 2\lambda y = 2 \\ 2\lambda z = -2 \\ x^2 + y^2 + z^2 = 1 \end{array} \right\}.$$

From the first three equations we get $x = \frac{1}{2\lambda}$, $y = \frac{2}{2\lambda}$, $z = \frac{-2}{2\lambda}$.

Substituting into the third equation we get $\frac{1}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{4}{4\lambda^2} = 1$. Hence $\frac{9}{4\lambda^2} = 1$ and so $\lambda = \pm \frac{3}{2}$.

If $\lambda = \frac{3}{2}$ then $x = \frac{1}{3}$, $y = \frac{2}{3}$ and $z = -\frac{2}{3}$. If $\lambda = -\frac{3}{2}$ then $x = -\frac{1}{3}$, $y = -\frac{2}{3}$ and $z = \frac{2}{3}$.

The maximum is 3 at $(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$. The minimum is -3 at $(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$.

Exercise 8: We wish to maximise $12x^{2/3}y^{1/2}$ subject to the constraint $4x + 3y = 7000$.

Let $F(x, y, \lambda) = 6x^{2/3}y^{1/2} - \lambda(4x + 3y - 7000)$.

$$\text{Then } \frac{\partial F}{\partial x} = 4x^{-1/3}y^{1/2} - 4\lambda, \quad \frac{\partial F}{\partial y} = 3x^{2/3}y^{-1/2} - 3\lambda.$$

We solve the system:

$$\left. \begin{array}{l} 4x^{-1/3}y^{1/2} = 4\lambda \\ 3x^{2/3}y^{-1/2} = 3\lambda \\ 4x + 3y = 7000 \end{array} \right\}.$$

Square the first equation:

$$16x^{-2/3}y = 16\lambda^2.$$

Now multiply by the second equation to eliminate x :

$$48y^{1/2} = 48\lambda^3. \text{ Hence } y^{1/2} = \lambda^3 \text{ and so } y = \lambda^6.$$

To eliminate y we simply multiply the first two equations in the original set:

$$12x^{1/3} = 12\lambda^2. \text{ This gives } x^{1/3} = \lambda^2 \text{ so } x = \lambda^6.$$

We now substitute $x = y = \lambda^6$ into the third equation to get $4\lambda^6 + 3\lambda^6 = 7000$.

Hence $7\lambda^6 = 7000$ and so $\lambda^6 = 1000$. Thus $x = y = 1000$. This gives $D \approx 37947.3319$. So the maximum number of doses that can be made, without spending more than \$1000 is about 38000.