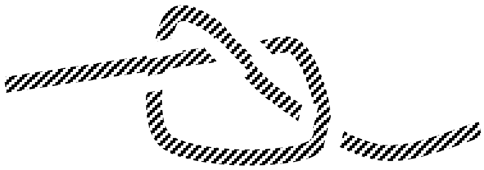
KNOTS

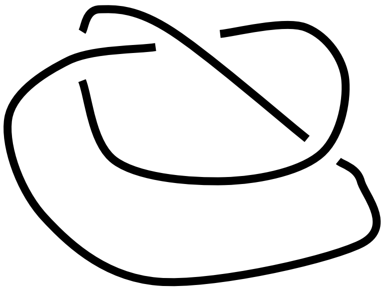
1. Knots

As every sailor knows, a knot is something you tie in a rope to hold something together.



And by weaving the ends in and out of the knot in the right way we can undo any knot. So if we work with ends to our piece of rope all knots are equivalent to a straight piece of rope.

But suppose we join the two ends together.



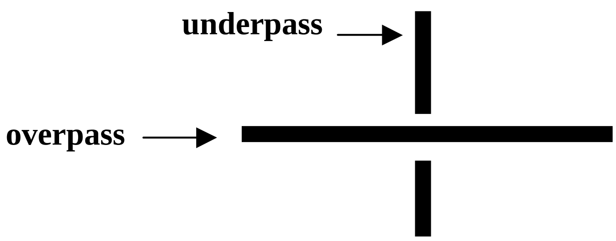
We can no longer untie the knot. We can still change its shape – even make it very much more complicated – but it will always be possible to manipulate it back to the way it was at the moment we joined the edges together. It is as if it retains the memory of what it was lie at that stage.

To a mathematician a **knot** is the path followed by a piece of rope that weaves in and out of itself, and has the two ends joined together. Mathematicians study knots as part of a geometric subject called "topology". There was a time when knots seemed important in chemistry, in connection with knotted molecules. By the time chemists had lost interest in them, mathematicians had become interested in them for their own sake. In more recent times physicists and biologists have dabbled with them. For example the way DNA is knotted may have some biological significance. But to mathematicians they are a fascinating study whether or not they will be ever be useful.

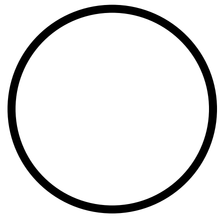
Knots are inherently a 3-dimensional phenomenon. But to study them we draw 2-dimensional pictures of them, as we have done. In such a picture, the places where the knot crosses over or under itself are called **crossings**. In the above picture there are three such crossings.

To indicate which pieces of the knot pass over a crossing, and which pass under, we use the obvious convention of putting a break in the portion that goes underneath. But remember – the knot really isn't broken at these points.

The portion of the knot that passes over the top is called the **overpass** and the other portion is called the **underpass**.



If you take a piece of rope and tie the ends together without first tying a knot, you get a circle. Well, it may not look like a circle, but if we are careful we can lay it out as a circle.

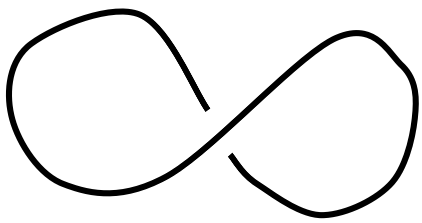


It would be quite understandable if you were to say that a circle isn't knotted. But mathematicians have a way of treating "nothing" as if it was something. For example many centuries ago the number zero wasn't considered to be a number. After all, if you have nothing, there's no need to count it. But the invention of "zero" was an important step in the development of numbers. Without it we wouldn't have our place-value system of notation. We'd have to write 2004 as MMIV, and you can imagine how much more difficult it would be to do long division in Roman numerals.

We call a circle the **unknot**. This might sound as though we are saying that it isn't a knot, but that's not so. It is a knot, though a very trivial one, and we describe its trivial nature by the word "unknot".

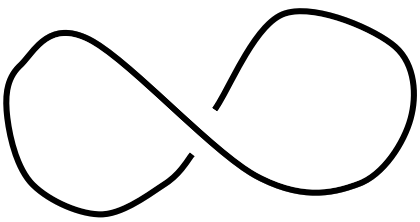
The unknot, when drawn as above, has no crossings. Clearly any knot with no crossings can be made to look like any other, so we say that there is just one knot that can be drawn with no crossings – the unknot.

What if we have just one crossing? Clearly any knot with one crossing can be deformed so that it looks like the following picture.



So there is just one knot that can be drawn with only one crossing. But if you look at it closely you will notice that we could untwist the right-hand loop and so remove that crossing. This knot is just the unknot, in disguise.

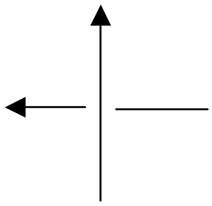
Another knot with one crossing is the following.



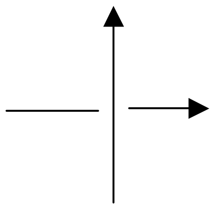
As 2-dimensional drawings these are quite different. Imagine that these pictures represented a one-way road that loops around and passes over itself at one point. As you drove over the top of the bridge (the overpass) and looked down you would see the traffic below. But in the first case the traffic would be coming under the bridge from your right while in the second case it would be coming in from your left.

Of course as 3-dimensional knots both are equivalent to the unknot, but in more complicated situations this distinction can be of real importance.

Suppose we give a knot a direction (it doesn't matter which of the two possible directions we choose. As we move around the knot in this direction we call a crossing a **positive crossing** if, when you pass over the overpass the underpass comes from the right. If it comes from the left, the crossing is a **negative crossing**.

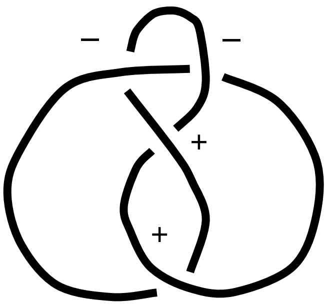


+ve crossing



–ve crossing

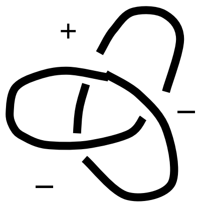
Example 1:



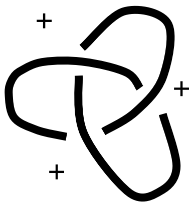
But, whether or not the crossing is positive or negative, a knot with one crossing is simply the unknot.

With two crossings we would have to consider two positive crossings, two negative crossings, and one of each. But in all cases a knot with two crossings is just the unknot. To get a non-trivial knot (one that isn't the unknot) we need at least three crossings.

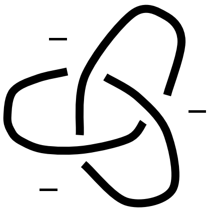
Some knots with three crossings are trivial. For example the following knot can be simply untwisted to produce the unknot.



However if we take the same picture, but make all three crossings positive crossings we get the following non-trivial knot.



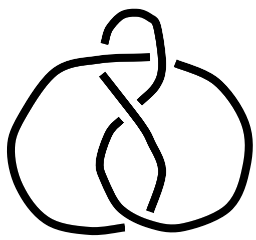
And if we make them all negative crossings we get a similar picture.



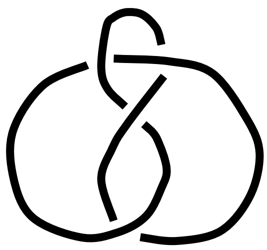
These knots are called the **trefoil knots** (the positive and negative trefoil knots, respectively). Now it is intuitively obvious that these are not equivalent to the unknot. There is no way the crossings can be eliminated, without breaking open the rope. Intuitively obvious, perhaps, but how could we prove it?

Somewhat less obvious is the fact that these two trefoil knots are distinct from one another. You can never transform a positive trefoil into a negative one or vice versa. Again there is the question as to how we can be sure of this.

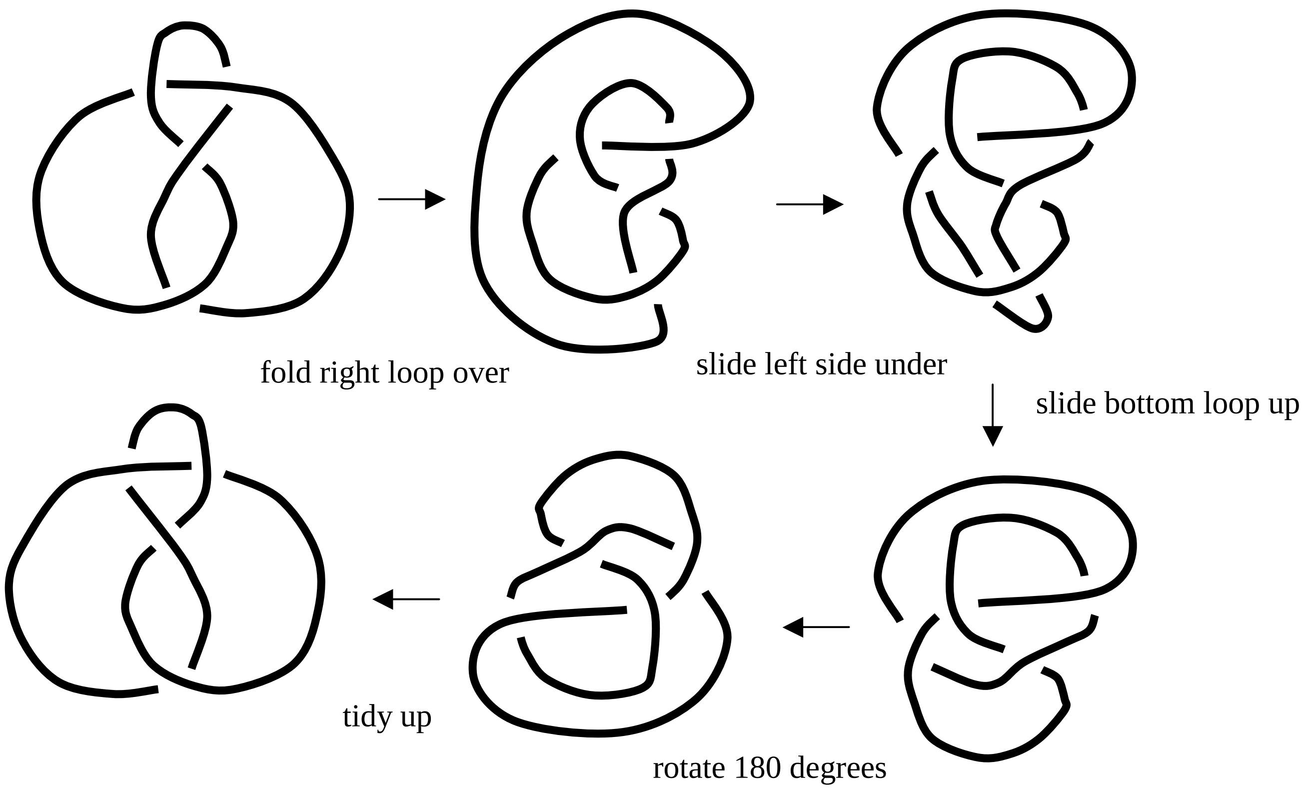
Two knots that are related to one another, by over crossings becoming under crossings, are called **conjugate knots**. In the case of the trefoil knots the two conjugates are different knots. But consider the following knot, with four crossings, called the **figure 8 knot**.



Its conjugate is the following.

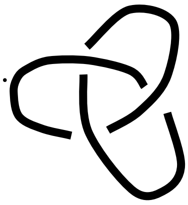


In this case it is possible to change the knot into its conjugate. Here is a sequence of steps showing how it can be done.



2. Colouring Knot Maps

Every picture of a knot in the plane gives us a map on the plane, with vertices, faces and edges. The vertices are the crossings, the edges are the sections of the knot that go from one crossing to the next, and the faces are the regions enclosed by the edges. We include the region outside the knot as one of the faces.



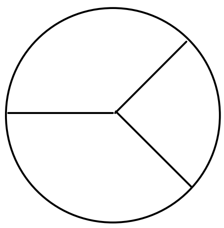
Example 2: The trefoil knot has V = 3 vertices, E = 6 edges and F = 5 faces.

Notice that E = 2V and F = V + 2. This is always the case.
The reason why there are twice as many edges as vertices is because as we proceed around the knot we alternately come to an edge and then a vertex and we visit each vertex twice, once on an overpass and once on an underpass.

Now for any map on a plane, where we count the outside as a face, Euler's formula gives V + F – E = 2. Since E = 2V this becomes V + F – 2V = 2 or F – V = 2. So F = V + 2.

Another interesting fact about the maps that arise from 2-dimensional pictures of knots is that we can colour the faces with two colours, black and white, in such a way that adjacent faces have different colours.

This is not always the case with maps. For example the map



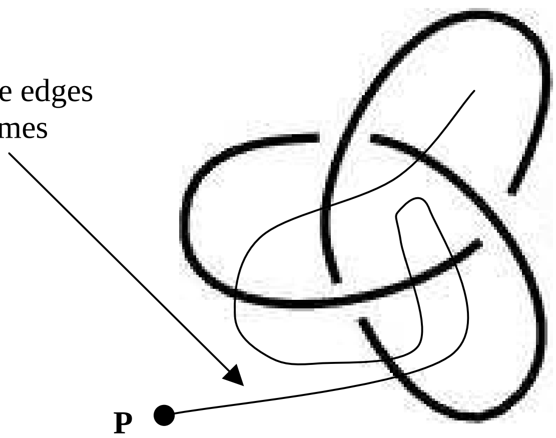
can't be coloured with just two colours, if adjacent faces are to have different colours. In fact if we include the outside face, we have four faces each of which is adjacent to the other three. W
For this map we would need four different colours.

But, of course, this map didn't come from a knot. Those maps that do come from a knot can be colours with just two colours.

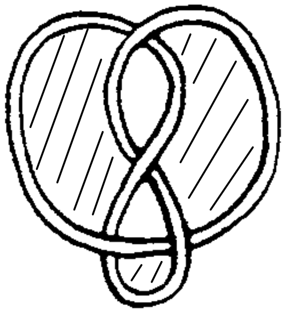
Colour the outside face white. Now take a point P outside the knot. To decide what colour to colour any face you take a path from P to the middle of that face. It can be any path, but it must avoid the crossings.

Now this path will cross a number of edges. If it crosses an edge an even number of times then you colour the face white. If it crosses an odd number of times you colour it black.

This path crosses the edges of the knot map 7 times



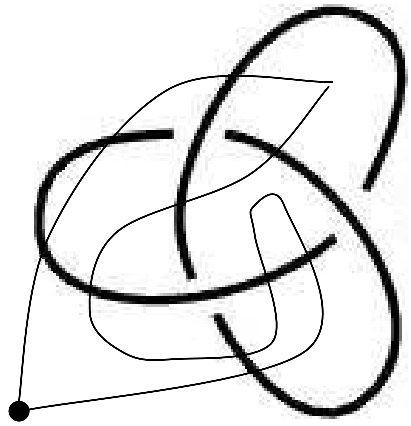
So here is how you would colour the figure 8 knot.



It is clear that adjacent faces would have different colours because as you cross from one face to an adjacent one you are crossing an edge one more time, and this changes an even number into an odd one or an odd one into an even one.

But before we can be satisfied with this explanation there is one potential difficulty. What would happen if there was one path from P into a certain face that crossed an edge an even number of times and another path that crossed an edge an odd number of times? We would have to colour the face both white and black, which would never do!

This situation can never arise, however. We can join up the two paths to enclose a certain region and, if one path crossed the knot an even number of times and the other an odd number then the knot would pass in and out of this region an odd number of times. But this can't be because as you move around the knot you must enter this region as many times as you exit it by the time you return to where you started.



One path crosses the knot 7 times and the other 3 times. Taking these two paths as enclosing a region the knot crosses this closed boundary 10 times altogether.

So although the number of times a path from P to a certain face may differ with different paths, it will either be always even or always odd, and so there is no difficulty in deciding which colour to use.

3. Equivalence of Knots

The question that engages the interest of knot theorists most is the fundamental question:

How do you decide whether two given knots are equivalent?

Remember that two knots are equivalent if you can change one into the other, without breaking the knot open.

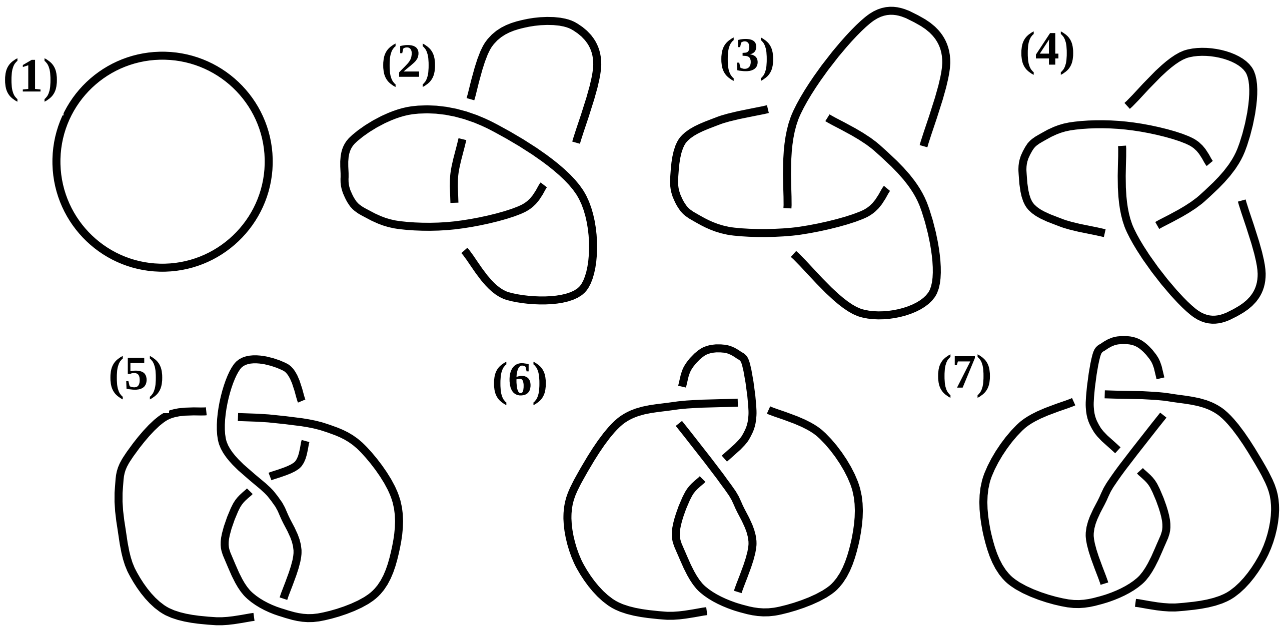
If two knots are equivalent there is no more convincing proof than to provide a recipe for actually doing it. We showed that the figure 8 knot and its conjugate are equivalent by showing how one can change one into the other.

But what if they are not equivalent? Do we take a piece of rope that's tied up in one way and try to change it into the other knot? If we can't do it does that prove it can't be done? Not at all! It proves one of three possibilities:

(1) the knots are not equivalent;
(2) we haven't tried long enough;
(3) we are not very clever at this sort of thing.

Our inability to do something is never a proof that something can't be done. How many times have you given up on a puzzle with a cry of frustration "it's impossible!" only to have somebody show you how it can be done?

Example 3: Here is an exercise to show you how difficult it can be to decide whether two given knots are equivalent. There are seven pictures here. What you have to do is, as best you can, sort them into groups so that those in the same group are equivalent while those in different groups are not equivalent. In some cases you may be able to see how to transform one into another and so you would put them in the same group. In other cases your intuition may convince you that two knots are probably equivalent, even if you can't see how it might be done.



If you just went on the basis of appearance you might sort them as follows:

(1)	**(2) (3) (4)**	**(5) (6) (7)**

But on closer inspection you would notice that (2) is the unknot in disguise. If you look closely at (5) you will perhaps be able to see that it can be transformed into (4). We have seen that (6) and (7) are equivalent and have been told that (3) and (4) are not. But what about (3) and (6) – one of the trefoils and the figure 8 knot? It doesn't seem likely that one can be transformed into the other. In fact you may be utterly convinced of this fact. And you would be right. But how do you *prove* that you are right?

In fact the 7 knots, sort into groups of equivalent knots are:

(1) (2)	**(3)**	**(4) (5)**	**(6) (7)**

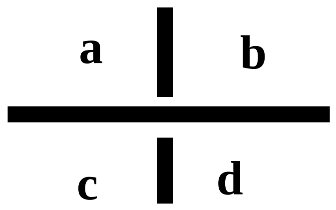
4. The Alexander Number of a Knot

The way we prove that two knots are not equivalent is to find a number that we can give any knot where if two knots are equivalent they have the same number. Then, if they have different numbers they cannot be equivalent.

Unfortunately the situation is not perfect. If they have the same number that doesn't prove that they *are* equivalent. But, still it's something. We will be able to show that in the above set of examples (1) and (2) have number 1, (3), (4) and (5) have number 3 while (6)and (7) have number 5. So this will be enough to prove that the trefoils are not equivalent to the figure 8 knot, but it won't be able to prove that the two trefoils are not equivalent. That requires much more powerful techniques.

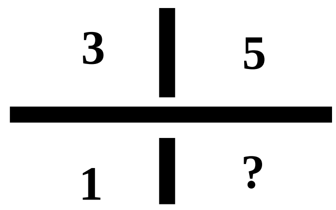
The number that we shall describe is called the **Alexander number** of the knot.

Take a 2-dimensional picture of the knot. We are going to write numbers into each of the faces, including the outside. We shall arrange for these numbers to “balance” at every crossing. This means that the sum of the two numbers on one side of the overpass must be the sum of the numbers on the other side.



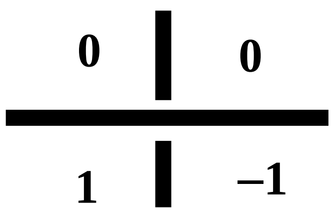
gives a + b = c+ d.

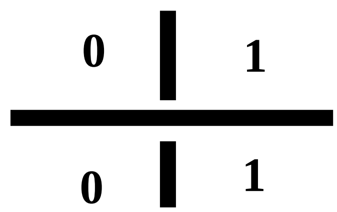
If three out of the four faces that surround a crossing have already been given numbers then this balancing equation enables us to find the fourth. For example, if we have:



then the number in the remaining face must be 7 since 3 + 5 = 1 + 7.

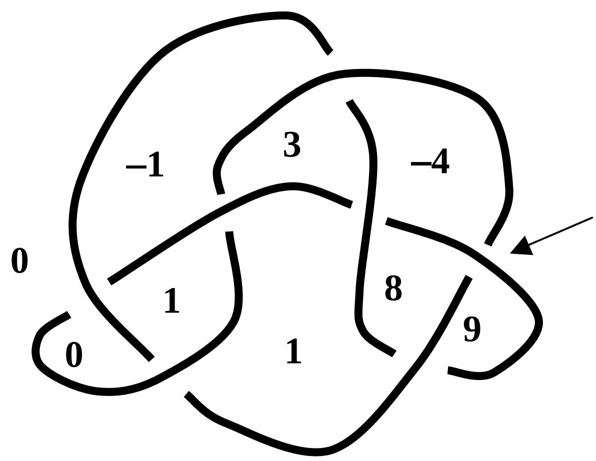
To get started we put 0, 0 and 1 in the three faces surrounding one of the crossings. In practice it is a good idea to put 0 in the outside face. The remaining face at this crossing will be given the number +1 or –1, whatever is needed to balance at that crossing.





We continue, wherever there are three out of the four numbered faces at a crossing by working out the value of the remaining face.

Example 4:

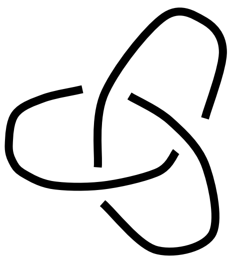


Here we get 17 = – 4 so
21 = 0.

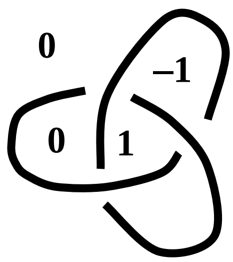
At the last crossing we appear to get a contradiction: 21 = 0. It would be a contradiction if this were ordinary arithmetic. But in arithmetic modulo 3 or 7 or 21 this is perfectly correct.

The Alexander number is the largest modulus that makes the last crossing balance. In this case, it is 21. So if this equation at the last crossing is n = 0 for some positive number, n, then that number is our Alexander number.

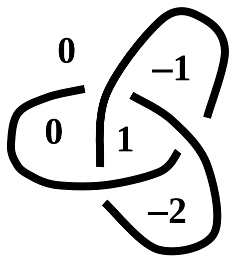
Example 5: Find the Alexander number of the negative trefoil.



Solution: We start by putting down our first balanced crossing.

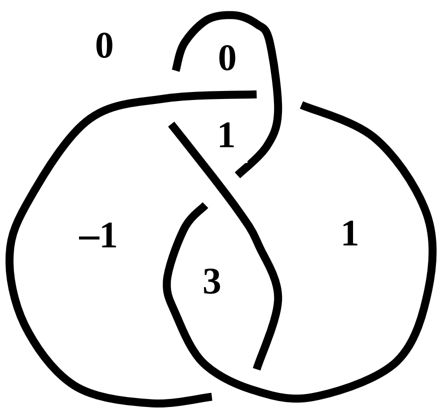


Now we balance the crossing on the right. (Remember that the whole of the outside is 0 so on one side we have –1 + 0 = –1.) So the remaining face must be –2 since –1 + 0 = 1 + (–2).



At the bottom crossing we've got 0 + 1 = –2 + 0 which gives 3 = 0. The largest modulus that makes this true is 3 so the Alexander number for this knot is 3.

Example 6: Find the Alexander number of the figure 8 knot.
Solution:

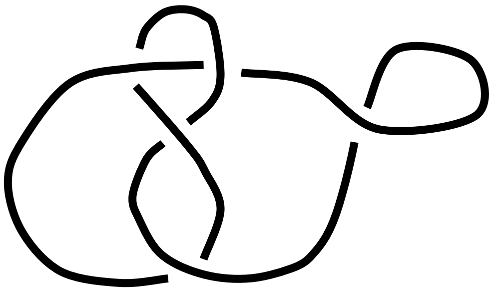


At the bottom crossing we get 3 + 1 = –1 + 0 which gives 5 = 0, so the Alexander number is 5.

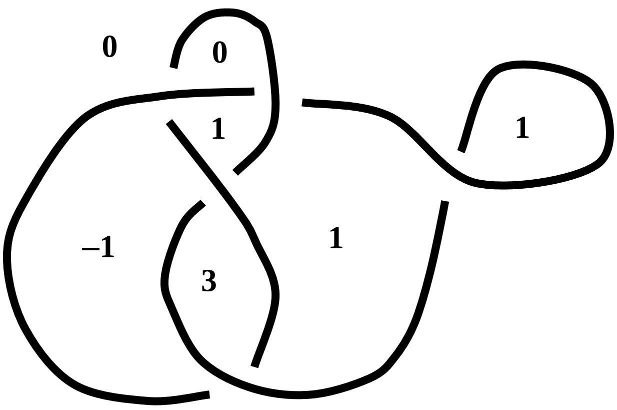
The Alexander number of the trefoil is 3 (actually both trefoils have Alexander number 3) and for the figure 8 it is 5. Since 3 is different to 5 this proves that the trefoil and the figure 8 knot are not equivalent. It is impossible to transform one into the other.

Of course, when we say "proves" you must remember that we haven't shown that this number will always be the same for the same knot, even if we change its appearance – even if we add extra crossings. We will not be giving a proof of this fact as it relies on an advanced area of mathematics called group theory. But this next example will provide some circumstantial evidence.

Example 7: Find the Alexander number of the following knot.

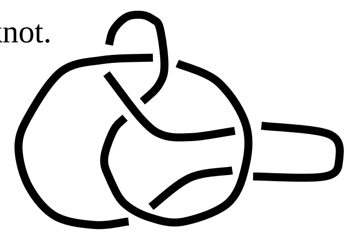


Solution:

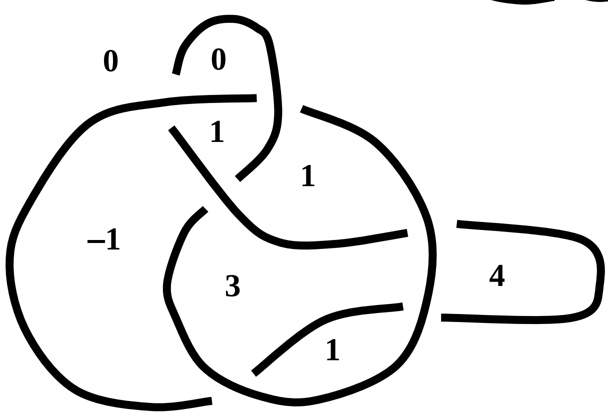


As before we get the equation 3 +1 = –1 + 0 at the final crossing and so the Alexander number remains as 5. Notice that introducing a "kink" adds an extra crossing and an extra face, but does not change the final equation.

Example 8: Find the Alexander number of the following knot.

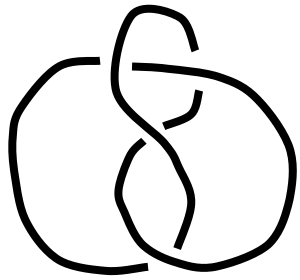


Solution:

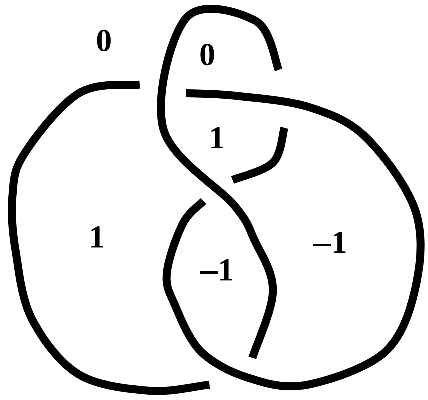


As before, at the bottom crossing we have the equation 3 + 1 = –1 + 0, giving the Alexander number of 5. Notice how pushing one section of the piece underneath another has broken one of the faces of the figure eight knot into two. But notice also that the balancing equations at the newly created crossings mean that both parts of the split face get the same number, so at the final crossing we must get the same equation.

Example 9: Find the Alexander number of the following knot.



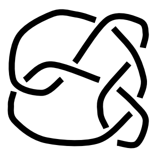
Solution:



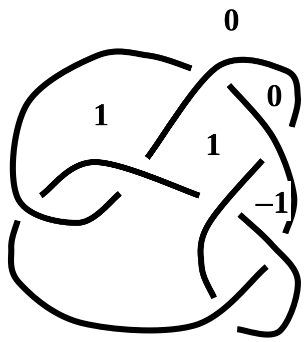
This time, at the bottom crossing, we have –2 = 1 so 3 = 0. The Alexander number is this knot is therefore 3. So, if you hadn't noticed, this isn't the figure 8 knot. In fact it's the one we saw earlier which looks a bit like the figure 8 knot but in fact can be transformed to one of the trefoil knots. And being a trefoil knot it has to have the trefoil's Alexander number, namely 3.

Or, putting it another way, even if you disguise the trefoil by making it look like a figure 8 knot you cannot fool the Alexander number. It remains the same.

Example 10: Find the Alexander number of the following knot.

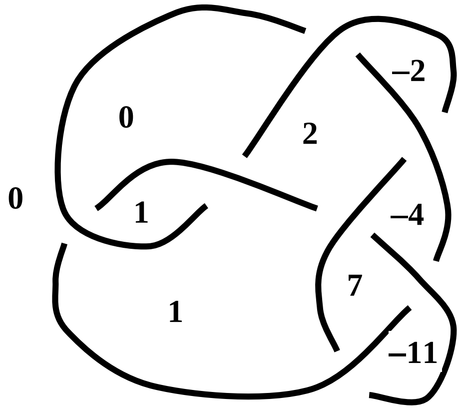


Solution:



At this point we come to a grinding halt since there is no crossing where we have three out the four faces numbered.

But if we proceed differently we can overcome this problem.



At the bottom crossing we have 1 + 7 = –11 + 0 So 19 = 0. The Alexander number is 19.

As you an see we can sometimes get around an impasse by being a bit more clever – but not always. There are more complicated knots where we cannot proceed, no matter what we do. It is still possible to find the Alexander number, even for the most complicated of knots, but this requires more advanced techniques that we are able to describe here.