

INTERLUDE: STORY

Pam and the Prime Minister

The pure voices of the boy sopranos floated up to the lofty recesses of St Mersennes's. "... primes without end ... a-men". The service was over.

Elizabeth turned to her friend and said, "isn't he a dream - those eyes."

But Pamela said, "I was more interested in his sermon. It seemed very persuasive but I'm sorry, Elizabeth, I'm still an agnostic. I just can't believe in your doctrine of the Infinitude of Primes. I mean, perhaps it *is* true that there are infinitely many prime numbers, I can't see how you could ever know for sure."

"But Pam, you can see here in 'Primes Ancient and Modern' and here in 'The Book of Common Primes' that there *are* primes for ever and ever into eternity. Look there's no sign of them petering out."

They had reached the church door and the young curate held out his hand.

"I trust you enjoyed the service, Elizabeth?", he greeted them, while looking at Pamela with his penetrating blue eyes.

"Oh yes", gushed Elizabeth, "I found the primes *so* inspiring. But I'm afraid my friend here is an unbeliever."

Pamela smiled sheepishly. "It's just that I can't see how you can be so certain. I admit that it seems very unlikely that the list of primes will ever come to an abrupt halt but ... I mean ... it *is* possible. After all primes become rarer as you go among the larger numbers. Is it inconceivable that there is a largest prime out there somewhere?"

She pointed vaguely in the direction of the churchyard, but there were too many people behind them waiting to shake the curate's hand to continue the conversation.

"How about if you and Elizabeth come to the rectory next Sunday afternoon. We could talk some more over tea and scones."

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Reverend Matthews poured the tea and passed round the excellent scones that Mrs Duffy had made.

"I'm sure Pam would like to believe that there are infinitely many primes but she doesn't seem to have enough faith."

"If only there was some way you could prove it to me," sighed Pamela, "but of course that's impossible. Even if I spent from now till the end of the world factorising numbers I'd only be considering a finite number of possibilities. There's no way the question can ever be settled."

"Well", said the curate, "you do believe that there are infinitely many numbers altogether don't you?"

"Oh yes, of course, that's obvious. I mean you just keep on adding one to get bigger and bigger numbers."

"And if I claimed that there was a biggest number?" he asked.

"Then I'd say 'what about that number plus one'".

"Exactly. I'd be forced to admit that my claim was false."

"But it's not the same with prime numbers", protested Pam. "Because all primes are odd, well except for two of course. And so the largest-prime- plus-one would be an even number so it couldn't be prime. And the next number after that mightn't be prime either."

At this Reverend Matthews took a handful cubes from the sugar bowl and laid them neatly in a row on the damask tablecloth. "Suppose", he said, "that each of these sugar cubes represents a prime number. Here's two and three, five, seven, eleven and thirteen, seventeen, nineteen and twenty-three. Now just suppose, for argument's sake, that there *does* exist a largest prime."

He scooped up the glistening white cubes and put them back into the sugar bowl. "Just suppose that this bowl contains every prime number up to the largest prime."

"Well, alright then," agreed Pamela, "just for the sake of argument. But don't forget that I maintain that believing in a largest prime is just as illogical as believing in the Infinitude of Primes. You'd need infinite time to prove it one way or the other.

"I hope it won't take *that* long", he said looking at his watch and smiling, "I have to conduct Evensong at six o'clock!" He picked up the pot containing all-the-primes-in-the-world and said, looking earnestly at Pam, "we have here every prime number that exists and, we're supposing, there are only finitely many of them."

"But a very large finite number", said Elizabeth helpfully.

"Now we can multiply all these numbers together to get an exceedingly large number."

"What if there's not enough paper in the whole world to write it down?", asked Elizabeth.

"That's of no consequence", he assured her. " We can conceive and discuss numbers bigger than the number of atoms in the cosmos. Don't forget, a number's existence doesn't depend on the vital statistics of our universe."

"But I don't see what you're getting at", said Pam as she took another scone. "The result of multiplying all the prime numbers won't be a prime number itself, so where's the contradiction, if there is one?"

"But would you agree that this product-of-all-primes will be divisible, exactly, by all prime numbers?"

"Yes Pam, don't you see," said Elizabeth excitedly, "every prime number will go into it exactly because every prime will be one of its factors!"

Pam did see. She was more concerned about where the argument would go from there.

"Well the product-of-all-primes will be divisible by all primes so the product-of-all-primes plus one can be divisible by *no* primes." Reverend Matthews leaned over towards Pamela to make sure she got the point.

"You mean because no two consecutive numbers can have a common factor?", said Pam thoughtfully.

"Exactly. So we're brought to a number which has *no* prime factors. Now this product-of-all-primes-plus-one is too big to be a prime itself."

He put his hand on Pam's arm to steady her from the impact of the contradiction that was to follow. "But every number, if not prime itself, can be factorised into prime factors, so it *must* be divisible by at least one prime and hence we reach a contradiction. And remember Pam, that contradiction only came about because we were foolish enough to contemplate a largest prime."

Pam appeared to recover quickly from the shock of the contradiction, if she felt it at all. But in case of an after shock his hand across the table continued to steady her arm.

Pam, in fact was so deep in thought that she forgot for a moment that she even had an arm. She screwed up her pretty, little nose, trying to make sense of it all. At last she

discovered the arm, drew it away from the young curate's grasp and picked up an unused sugar cube which had lain unnoticed on her saucer.

"Well all that means," she said, "is that since it isn't divisible by any of the primes already in our pot it must itself be prime, one we overlooked. So we just pop this extra prime in the pot." And she dropped the sugar cube into the bowl.

"But", protested Reverend Matthews, "you'll just get the same contradiction all over again."

Pamela picked up a handful of the cubes and dropped them one by one back into the bowl. "So, as fast as you keep getting a contradiction I just keep adding more and more primes to the pot. I can always keep one step ahead of you." She grinned, confident that she had him beaten.

But Reverend Matthews yielded no ground. In fact he must have been about to deliver another intellectual earthquake because he felt the need to steady her arm again. "The point is Pam, you agreed that we had *all* primes in our finite pot, and now that I contradict you, you want to add another. That's hardly fair."

But Elizabeth came to her rescue. "Is it such a sin to change ones mind?"

"Look if it was a game of chess I'd be only too glad to let her change her mind to correct an oversight. But she can't claim to be always one step ahead of me just because I let her keep changing her move every time she lands in trouble. Besides, finding a prime that's not in the prime pot isn't an oversight. We agreed to *define* the contents of the pot to consist of all primes. It's just an inescapable contradiction. And any assumption that leads to a contradiction must be false. Q.E.D."

He appeared to think that this final blow would require a little extra support and it didn't matter that he upset the sugar bowl in the process because those little crystal cubes had served their purpose. Pamela displayed her discomfort at her intellectual position by blushing brightly. She looked down at the spilt cubes on the table as if pleading with them to deliver her the inescapable conclusion of the argument.

At last she looked up into Reverend Matthews deep blue eyes and sighed, "I suppose you're right."

But Elizabeth, who had becoming more and more agitated while all this was going on, said tersely, "I'm not so sure now. If believing in the Infinitude of Primes stops people from changing their minds I think I'd rather be an agnostic!"