3. THE UNCOUNTABLE

1. Kindergarten Counting

One of the very first notions we came across in Mathematics was that of counting. We were shown pictures of familiar objects repeated several times and we learnt to associate a number with the collection.

For example, perhaps we were shown a picture of five ducklings with the symbol "5" underneath. We may have thought in our innocence that "five" was the name of these creatures, or that it was their colour, or in some way had something to do with what they were. But gradually, as we were shown five chairs, five pencils and so on, all associated with a "5", we began to acquire the idea of "fiveness". Soon we learnt to abstract the property of number from collections of all sizes, and then we began to count everything in sight.

In later years we learnt to do things with numbers. We learnt to add, subtract, multiply and divide. We learnt to solve problems using numbers (arithmetic) and later to solve problems using symbols to represent numbers (algebra). At the same time we learnt of different types of numbers — fractions, decimals and so on.

But now, let us start right back at kindergarten level and take a more mature look at the process of counting. When we learnt to count, we didn't sit back and reflect on what we were doing. We just did it. Most people never revisit their counting days to consider what counting really is. It is important to do so because shortly we will learn to count with infinite numbers and to do this we must have the right way of thinking about finite numbers.

2. The Same-number Balance

One might think that counting is the most fundamental concept in all of mathematics. And yet it is a complex idea built on an even more fundamental one.

Imagine walking into a completely full lecture theatre and looking around briefly. You are able to see at a glance that the number of people sitting down is exactly equal to the number of occupied seats. Is it because you quickly counted the people: 1, 2, 3, ... and then counted occupied seats and, lo and behold, you got 237 each time? No, that would have taken some time. It's really much simpler than that.

You just notice that nobody is sitting on someone else's lap, nobody is sprawled out over two or more seats and no bears, tigers or other non-people are occupying a seat. In an instant you notice that every person seated has a corresponding seat and every occupied seat has a person sitting on it. There is a pairing off with people and occupied seats, and it is the existence of this one-to-one correspondence that establishes the fact that there is the same number of seated people as occupied seats.

If you wanted the actual number of each you would have had to do something a little more complicated. You would have had to count.

The situation here is very similar to that with weights. An old-fashioned beam balance is a simple device for comparing the weights of two objects. By itself it cannot weigh things absolutely. It merely shows you whether or not the weights are equal. Pairing off in a one-to-one correspondence is the balance we use for counting.

**Two sets have the same-number-as each other if
it is possible to pair the elements of one exactly with
the elements of the other exactly.**

The reason for the hyphens in "same-number-as" is because it is a single concept, like "balance". As yet we haven't yet given an independent meaning to the word "number". Once we do, we will be able to identify "same-number-as" with "same number as" in the sense of each set having a number and those numbers being equal.

3. Standard Sets

A beam balance can only be used for weighing things absolutely, as distinct from comparing weights, if we have a set of standard weights. We need some 1 gram weights and 5 gram weights, and so on, perhaps up to 1 kilogram weights. We put combinations of these into one pan of the scales until they balance exactly the unknown weight. This enables us to associate a number with the object that we call its weight.

Before we can count, that is, associate a number with a set to represent its size, we need some standard sets to use in the comparisons. In kindergarten we were introduced to a system of symbols 1, 2, 3, ... and associated words. These "objects" were initially meaningless objects that had a defined ordering. "Two" comes after "one" and then comes "three" and so on. We learnt to recite this list "one", "two", "three" as we would a nursery rhyme.

What we were setting up in our brains was a nested collection of standard sets (each fitting inside the other for convenience) by just stopping at different places. These standard sets are:

STANDARD SET	**SIZE**
{ } (empty set)	0
{1}	1
{1, 2}	2
{1, 2, 3}	3
{1, 2, 3, 4}	4
....	..

What we are doing when we count a set is to select a standard set which pairs off exactly with it. The size of the set is just the number associated with it. (For finite sets it is the last symbol in the list but when we come to infinite standard sets we will need to invent new symbols.)

Perhaps as adults we learnt to count in sophisticated ways, grouping things together for convenience. But if we go back to the primitive act of kindergarten we point to each object in turn and call out the next number in the sequence. The last number we reach will automatically be the answer to the counting.

It's important we get it quite clear what the act of counting really means before we introduce our first infinite number.

To find the number of elements in a set:
find a standard set which can be put in
one-to-one correspondence with it.
The associated number is the answer.

4. The Smallest Infinite Number $\aleph_0$

Are you ready for your first infinite number? We need a standard set and then a symbol to represent its size. What better standard set than the set of all finite numbers {1,2,3,....}?

Now for a symbol. You see, we can't use the last element in the list because there isn't one. We could have used the standard "infinity symbol", ∞, but that would suggest that this is the only infinite number we're going to get. Besides it's not the symbol used by Georg Cantor who first investigated infinite counting around the end of the nineteenth century. He chose the first letter of the Hebrew alphabet, ℵ, and because it was the smallest infinite number he added the subscript "0". So our list of standard sets has been extended to the following:

STANDARD SET	**SIZE**
{ } (empty set)	0
{1}	1
{1, 2}	2
{1, 2, 3}	3
....	..
{1, 2, 3, 4, 5, 6, ...}	$\aleph_0$

5. In Search of a Bigger Infinite Number (Adding)

Now we begin our long journey, in search of an infinite number bigger than $\aleph_0$. With finite numbers we were always able to get a bigger number by adding one.

"My dad's played footy a trillion, trillion times!"

"My dad's played it trillion trillion plus one!"

Let's see if $\aleph_0 + 1$ is a bigger number than $\aleph_0$. Well it's certainly not smaller. But could it be no bigger?

Before we can answer that we must say what we mean to add one to a number, in a way that makes sense for infinite numbers.

When we were learning how to add such finite numbers as 2 and 3 we possibly had a picture of two ducks and three rabbits. Count the ducks. Two. Count the rabbits. Three. How many animals altogether. Before we learnt to add we would have had to count the entire menagerie. One, two, three, four, five. The whole collection of animals matches exactly with our standard set {1, 2, 3, 4, 5} and so its size is 5. We have demonstrated that 2 + 3 = 5.

As time went on we learnt ways of adding without counting. But if pressed for what it means for 37 plus 63 to equal 100 we would have to say something like: "if you take 37 of one type of thing and combine it with 63 of something else we get 100 things altogether.

Addition corresponds to combining two sets of things together. But it is important that the two sets have nothing in common, otherwise we are double counting. So here is our definition of the sum of two numbers.

To add the numbers m and n:

(1) Take a set of size m;
(2) Take a set of size n;
(3) Ensure that these sets are disjoint (have no common elements);
(4) Combine them into one set (take the union of these disjoint sets);
(5) Put this union into 1-1 correspondence with a standard set.

(6) The number of elements in this combined set is defined to be m + n.

Let's use this to calculate $\aleph_0 + 1$.
Take a set of size $\aleph_0$. The standard set {1, 2, 3, ...} will do.
Now a set of size 1. The standard set of size 1 is (1}..
But these two sets have "1" in common. So change the second set to {0}.
The union of these two sets is {0, 1, 2, 3, ...}.

Now this certainly appears to be bigger than the set {1, 2, 3, ...} but is it? No. We can match {0, 1, 2, 3, ...} off exactly with {1, 2, 3, ...}. Just write out these sets in rows and each number in the top row pairs off exactly with the one below it:

0 1 2 3 4 5 ...
1 2 3 4 5 6 ...

Since neither set has a last element, there is nothing in one row without a mate in the other. According to our definition, therefore, these two sets have the same number of elements. In other words $\aleph_0 + 1 = \aleph_0$.

"But that's absurd. If you add something extra of course you make it bigger!" Careful, you're revealing your parochialism. It's just like someone who's lived all his life in some small outback country town. "Of course, if you go into a bank they'll know your name."

You're no longer in the finite backwoods you've been in all your life. This is the big city of the infinite. Some facts you've accepted as having universal application, you now find are just curiosities that only work for finite numbers. Other things you've learnt *do* extend into the infinite.

What can you trust in this strange new world? Just the definitions and your logic.

So, contrary to naive intuition, you don't make an infinite number bigger by adding one to it. Our search for a number bigger than $\aleph_0$ has so far failed. What about $\aleph_0 + \aleph_0$?

For this we need two disjoint sets of size $\aleph_0$. The standard set {1, 2, 3, ...} will do for one of them and we can take the negative numbers for the other: {−1, −2, −3, ...}. We can set these out in a table with two infinite rows:

1	2	3	4	5	...
−1	−2	−3	−4	−5	...

Surely these can't be paired off with our standard set for $\aleph_0$! To do that we would have to squeeze both infinite lists into a single one.

But that's not difficult. Simply take from each row alternately:
1, −1, 2, −2, 3, −3, ...

Nothing is left out, but now that they are in a single infinite list we can pair them off with our standard set {1, 2, 3, ...}.

1	−1	2	−2	3	−3	...
1	2	3	4	5	6	...

Note that any infinite set which can be listed in a single list has size $\aleph_0$. We just pair the first thing in the list with 1, the second with 2, and so on. Another word which is used for this is *countable*. A *set* is countable if its elements can be listed. Countable sets include the finite ones, as well as these sets which can be put in an infinite list. Our goal is to find an uncountable set, whose size will therefore be bigger than $\aleph_0$. And so far we have failed.

6. In Search of a Bigger Infinite Number (Multiplication)

We've not yet been successful in finding a number bigger than $\aleph_0$. But we were only using addition up till now. A much more powerful operation is multiplication. Perhaps $\aleph_0 \times \aleph_0$ is bigger.

But what do we mean by multiplication? Repeated addition? But that won't work with infinite numbers for it would mean that $\aleph_0 \times \aleph_0$ is $\aleph_0 + \aleph_0 + ...$ with infinitely many terms. Instead we use the idea of ordered pairs.

A table with 5 rows and 7 columns has 35 cells. Each cell corresponds to a pair (r, c) where r is the number of the row and c is the number of the column in which it lies. It's an ordered pair, that is, for example, (3, 5) is different to (5, 3) because they refer to different cells. So here is the basis for a recipe for multiplying infinite numbers.

To multiply two numbers m and n

(1) Take a set of size m;
(2) Take a set of size n;
(3) Form the set of all ordered pairs, with the first item in the pair coming from the first set and the second coming from the second set;
(5) Put this union into 1-1 correspondence with a standard set.
(6) The number of elements in this combined set is defined to be m × n.

Let's use it to find 2×3 and see if we get the answer 6.
Take a set of size 2. Why not the standard set {1, 2}.
Now take a set of size 3. Why not the standard set for 3 {1, 2, 3}.
These sets are not disjoint, but that doesn't matter for multiplication. The ordered-ness of the pairs will keep them apart.
Now take all ordered pairs with the first item in each pair coming from {1, 2} and the second from {1, 2, 3}. Here they are:

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)

and as you can see there are 6 of them. So we have proved, using our definition of multiplication, that $2 \times 3 = 6$. Which is just as well! Our extended definition of multiplication agrees with they way we've always multiplied numbers *but* it gives us a way of multiplying infinite numbers.

Now before we tackle $\aleph_0 \times \aleph_0$, let's first try $2 \times \aleph_0$.
Now take a set of size 2. The standard set {1, 2} will do but for a change we'll take {+, −}.
Take a set of size $\aleph_0$. The standard set {1, 2, 3, ...} will do.
The pairs (x, y) where x is a "+" or a "−" and y is in {1, 2, 3, ...} can be put in a table as follows:

(+, 1)	(+, 2)	(+, 3)	...
(–, 1)	(–, 2)	(–, 3)	...

Obviously this is very little different to what we had above and so $2 \times \aleph_0 = \aleph_0 + \aleph_0 = \aleph_0$ as we would expect. So we have not yet broken the $\aleph_0$ barrier. But we still have $\aleph_0 \times \aleph_0$ up our sleeve!

Take two sets of size $\aleph_0$. Since they don't have to be disjoint we may as well take the standard set {1, 2, 3, ...} for both. Now form all ordered pairs. These can be set out in a two-way infinite table:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	...
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	...
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	...
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	...
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	...
...	...	...	...	...	...

Can we squeeze this into a single infinite list? All we have to do is to list the diagonals, starting in the top left-hand corner:
First comes (1, 1),
then (2, 1) and (1, 2),
and then (3, 1), (2, 2), (1, 3)
and so on. Stretching them out into as ingle infinite list we get:
(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), (4, 1), (3, 2), (2, 3), (1, 4), (5, 1), (5, 2), ...

$\aleph_0 \times \aleph_0$ elements, all written in a single list, means that $\aleph_0 \times \aleph_0 = \aleph_0$. We still have not succeeded in finding a number bigger than $\aleph_0$.

Notice, by the way that fractions can be represented by pairs of whole numbers so the above diagonal process would give us a way of listing all fractions. So while there appear to be more rational numbers, in fact the rational number set has size $\aleph_0$.

7. The Search for a Bigger Infinite Number (Powers)

If we cannot find a number bigger than $\aleph_0$ we've made a lot of fuss for nothing. But in fact we are just about to reach our quest.

Raising numbers to powers is much more powerful an operation than either addition or multiplication. For example 10 + 10 = 20, 10 × 10 = 100, but 10^{10} = 10000000000.

You might like to try $\aleph_0^{\aleph_0}$, but instead we'll settle for $2^{\aleph_0}$, which is easier to discuss and is just as big.

How can we give a meaning to 2^n for a number n. Multiplying 2 by itself n times is satisfactory for finite n but not if n is infinite. The secret to the correct definition lies in the concept of subsets.

One set is a subset of another if everything in the first set is an element, or member of the second set. For example the set of all women in the world is a subset of the set of all people.

We allow a set to be a subset of itself, so the set of all people is another subset of the set of all people. We even include the empty set as a subset. The set of all people who have lived to be 1000 years old is a subset of the set of all people. It's just that it happens to be empty.

Take a set with two elements, say {1, 2}. How many subsets does it have? Well, what are the subsets of {1, 2}?

First there is the empty set {}. then the subsets {1}, and {2}, and finally the set itself {1. 2}. There are 4 subsets. This will be true of any set with 2 elements.

Take a set with 3 elements such as {1, 2, 3}. What are the subsets? They are: {}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3} and finally {1, 2, 3} itself. How many? Eight.

So a set with 2 elements has 4 subsets, a set with 3 elements has 8. Is there a pattern? Yes, a set with n elements has 2^n different subsets, at least we can say that if n is finite.

A quick way to see this is to consider that each subset corresponds to a decision for each element whether or not it is to be in the subset. Imagine a sergeant lining up his men and asking for volunteers for latrine duty. In true military fashion it is the sergeant who does the volunteering. As he goes along the row of men he says, "you're in, not you, nor you, yes I want you, non, non, no, yes, ...}.

There are n choices, each a choice from two alternatives, so altogether there are 2^n possible subsets. Now this, which is a fact for finite numbers, can be taken as the definition of 2^n for infinite numbers.

To raise 2 to the power n

(1) Take a set of size n;
(2) Form the set of all its subsets;
(3) Put this union into 1-1 correspondence with a standard set.
(4) The number of elements in this combined set is defined to be 2^n.

8. $2^{\aleph_0}$ is bigger than $\aleph_0$

Powers of 2 grow ever so quickly and so it is a simple fact that for finite n 2^n is bigger than n, usually very much bigger. But we shall show, by a very cunning argument, that 2^n is bigger than n for all numbers, n, finite or infinite.

Showing that 2^n is bigger than n involves two steps, and in particular that $2^{\aleph_0}$ is bigger than $\aleph_0$.

$2^{\aleph_0} \geq \aleph_0$

"At least as big as" means finding a way of pairing off all the elements of a set with some of its subsets. This is easy. You just pair off each element with the corresponding set with one element.

The elements of {1, 2, 3} can be paired off with some of its subsets, namely 1 ↔ {1}, 2 ↔ {2}, 3 ↔ {3}. The fact that there are subsets left over, such as {1, 2} etc, shows that 2^n is bigger than n, for finite n, but, as we have seen, having things left over after a pairing doesn't necessarily mean "bigger" because there could be another pairing that leaves nothing over.

$2^{\aleph_0} \neq \aleph_0$

The proof that $2^{\aleph_0}$ and $\aleph_0$ are different runs along very familiar lines. We suppose that $2^{\aleph_0} = \aleph_0$ and get a contradiction. So suppose then that $2^{\aleph_0} = \aleph_0$.

Let **N** be the set {1, 2, 3, ...}. This has size $\aleph_0$. To say that $2^{\aleph_0} = \aleph_0$ means that there must be a exact pairing off of the elements of **N** with the subsets. Every element has a corresponding subset and vice versa.

Now for a given number n, one of two things will be true. Perhaps n belongs to the subset that it corresponds to, and on the other hand perhaps not.

For example one of the subsets of **N** will be **N** itself, and of course the corresponding element belongs to it. At the other end of the scale, one of the subsets is the empty set and the corresponding element will not belong to it.

Suppose we call those elements which belong to the subset they correspond to, *internal* elements. Those which lie outside their corresponding subset will be called *external* elements.

In symbols, if we denote the subset that corresponds to the element x by f(x), and use the symbol "$\in$" to denote "is a member of" and "$\notin$" to denote "is not a member of", then we can describe these properties of being internal and external as follows:

x is *internal* if $x \in f(x)$
x is *external* if $x \notin f(x)$

Of course whether an element is internal or external would depend on the particular 1-1 correspondence. But if somebody claimed to have a way of pairing off all the elements of a set with all of its subsets (rash claim!) it is perfectly reasonable for us to expect that they could tell us whether any given element is internal or external.

Suppose, for argument sake, that somebody claimed to have paired off all the elements of {1, 2, 3, ...} with all of its subsets. Then, in principle, they must have a list such as the following:

1	$\leftrightarrow$	{11, 32, 117}
2	$\leftrightarrow$	set of powers of 2
3	$\leftrightarrow$	empty set
4	$\leftrightarrow$	set of all multiples of 3
5	$\leftrightarrow$	set of prime numbers
...	...	....
3427	$\leftrightarrow$	set of all numbers
...	...	....
185367	$\leftrightarrow$	set of even numbers
...	...	....
6738679	$\leftrightarrow$	set of all external numbers
...	...	....

If this was indeed such a list then 1, 3, 4 and 185367 would be external. They lie outside their corresponding set. The elements 2, 5, 3427 would be internal.

If somebody claimed to have such a list it would also be reasonable, in principle, for us to ask for the set of all external elements, or at least what element the set of all external elements corresponds to. There is such a subset and so if the pairing is exact, as claimed, there is a corresponding element. In the above example we are supposing that it is 6738679.

Is 6738679 itself an internal number or an external one. It has to be one or the other. If it is internal then it belongs to the set that it corresponds to, that is, it belongs to the set of all external numbers which would make it external. That's nonsense. If it is internal, it is external. So it can't be internal.

But wait! If it is external, it is a member of the set of external numbers. So it does belong to the set it corresponds to. But this would make it internal! That's nonsense too!

If it is internal then it is external. If it is external, it is internal. One big resounding contradiction! And that contradiction all rests on the assumption that we started with, that the elements could be paired off with the subsets. Therefore they can't be. That is the number of elements of any set cannot be paired off exactly with the elements.

This argument can be used for any set.

The elements of a set cannot be paired exactly with the subsets.

Suppose the elements of a set are paired off exactly with its subsets.
Let f(x) denote the subset that corresponds to x.
Let Y be the set of all x such that $x \notin f(x)$.
Let y be the corresponding element, so that $f(y) = Y$.
If $y \in Y$ then $y \notin f(y)$, so $y \notin Y$.
And if $y \notin Y$ then $y \notin f(y)$ and so $y \in Y$.
This is a contradiction, which tells us that such a 1-1 pairing is impossible.

9. The Universe of Infinite Numbers

So $2^{\aleph_0}$ is bigger than $\aleph_0$. We call it $\aleph_1$. Actually $\aleph_1$ is usually defined to mean the next infinite number after $\aleph_0$. But nobody knows whether that is $2^{\aleph_0}$ or not. So it seems reasonable to define $\aleph_1$ to be $2^{\aleph_0}$. But if we do that, what if somebody finds an infinite number between $\aleph_0$ and $2^{\aleph_0}$. We'd then have to call it $\aleph_{1/2}$ or something. Relax! That will never happen. Nobody will ever find any numbers between $\aleph_0$ and $2^{\aleph_0}$. How can we be so sure? Because it has been proved that the existence of something between the two is unprovable. Well surely that means there aren't any! Not so, because nobody has been able to prove that the next number after $\aleph_0$ is $2^{\aleph_0}$. What is more, nobody ever will because a proof exists that shows that it is impossible to prove the next number after $\aleph_0$ is $2^{\aleph_0}$!

Amazing stuff, but all quite logical. We can prove that the statement *there is no number between $\aleph_0$ and $2^{\aleph_0}$* can never be proved. We can also prove that the statement *there **is** a number between $\aleph_0$ and $2^{\aleph_0}$* can never be proved. The question is undecidable.

The statement that nothing exists between $\aleph_0$ and $2^{\aleph_0}$ is called the Continuum Hypothesis. It is a hypothesis, not a fact. But it isn't a conjecture which will be settled one day. It will forever remain an hypothesis. You could say that whether it is true or not is a matter of faith.

"I believe in the Continuum Hypothesis" your creed might run. Fine. That's perfectly consistent with everything else we know about mathematics. But the opposite view is equally logical. I suppose the proper stance to take would be that of an agnostic.

On the other hand, even though it can never be proved, there is a metalogical argument in favour of believing in the Continuum Hypothesis. Since nobody will ever find an actual example of a number between the two (for if they did the matter would be decidable, contradicting the proof of the matter's undecidability) then for all practical purposes there *isn't* one. Though this falls short of an actual proof of non-existence, it seems a reasonable position to take.

So taking $\aleph_1$ to be $2^{\aleph_0}$ we can then use the same argument as above to show that $2^{\aleph_1}$ is bigger than $\aleph_1$ and so on. That means there is a whole infinity of infinite numbers: $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, $\aleph_4$, ... each bigger than the one before.

So if we set out to construct a catalogue of numbers we would start with two rows in our table:

$$0, \quad 1, \quad 2, \quad 3, \quad 4, ...$$
$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, ...$$

But as, they say in the TV advertisements for steaks knives, "there's more!" If you take a whole collection of sets, one for each of the infinite numbers in the second row of this table, the size of that collection will be at least as big as any number in the row, and hence must actually be bigger than anything in the row (think about it!). This will then give us a number bigger than anything in the two rows, so we can use it to start a third row.

But then by taking successive powers of 2 we can work our way along the third row. Three infinite sequences of numbers. But wait, there's more. In the same way we got from the second row to the third we can get from the third to a fourth row, and a fifth and so on.

So our catalogue of numbers, all but the first row being infinite, now covers an entire infinite page, with infinitely many infinite rows. But there's still more. There exists a number bigger than any number on this page and so we can start a second page, and a third, and so on until our catalogue occupies infinitely many pages, each with infinitely many infinite rows.

But why stop at one such volume. There are infinitely many volumes on an infinitely long shelf, and infinitely many such shelves .... The human mind is a wonderful thing to be able to conceive, and even think logically about, such expansive concepts.

Is there any practical use to all this? Such a question brings us back to earth with a jolt. The answer is yes. Mathematicians have a real use for knowing about $\aleph_0$, $\aleph_1$ and to some extent about $\aleph_2$. We could live without most of the others. The number $\aleph_1$ is the number of points on a line, or the number of real numbers. The number $\aleph_2$ is the number of functions from the set of real numbers to itself.

Oh, and where does ${\aleph_0}^{\aleph_0}$ fit into all this? Well it's not really any bigger than $2^{\aleph_0}$.